Florian Cajori

ELEMENTARY

SYNTHETIC GEOMETRY.

BY

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PREFACE.

My conception of Comparative Geometry dates back to 1877, when, having had the good fortune in Berlin to meet a copy of Lobatschewsky, then rare and unappreciated, I was enjoying a comparison of it with my beloved Euclid.

I have at last mustered courage to put my Pure Spherics where it belongs. But as my first book is geometry with one parallel geodesic, my second book geometry with no parallel geodesic, my third book should be geometry with more than one parallel geodesic through the same point (the Lobatschewsky-Bolyai elementary geometry). So it shall be, if, after another decade, I write still another geometry.

George Bruce Halsted.

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TABLE OF SYMBOLS.

bi'	bisector.
e.g.	exempli gratia [for example].
p't	point.
p'ts.	points.
quad'.	quadrilateral.
r't	right.
r't bi'.	right bisector [perpendicular bisector].
sq'	square.
s't	straight.
s'ts	straights.
$\odot$	circle.
$\bullet$	surface of circle.
$\odot^s$	circles.
$\triangle$	triangle.
$\triangle^s$	triangles.
$\overset{\frown}{\triangle}$	spherical triangle.
$\therefore$	therefore.
$\sim$	similar.
$\sim C$	center of similitude.
$=$	equivalent.
$\cong$	congruent.
	symcentral.
$\cdot\!\mid\!\cdot$	symmetrical.
$\parallel$	parallel.
$\parallel^s$	parallels.
$\parallel$ g'm	parallelogram.
$\perp$	perpendicular.
$\perp^s$	perpendiculars.
$+$	plus.
$-$	minus.
$<$	less than.
$>$	greater than.
$\measuredangle$	angle.
$\measuredangle ABC$	angle from ray BA to ray BC.
$\measuredangle ab$	angle from ray a to ray b.
$\odot C(r)$	circle with center C and radius r.
Λ	perspective.
ΛC	center of perspective.
$\overline{\wedge}$	projective.

ELEMENTARY SYNTHETIC GEOMETRY.

BOOK I.

SYMMETRY, SYMCENTRY, AND CONGRUENCE.

CHAPTER I.

THE PRIMARY CONCEPTS OF GEOMETRY.

1. A natural object, say a crystal, is bounded; and this boundary divides it from the air around it, but is not a thin film of the crystal itself. It is where the crystal ends and the air begins. It is also a boundary of the air where it joins the crystal, but it is not air. It is the boundary between the two, and is common to the crystal and the air.

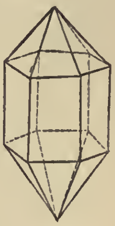


Fig. 1.

2. A boundary of the sort capable of wholly enclosing a solid, so that nothing could get into the solid except through this boundary, but itself no solid, is called a *surface.*

3. Surface is an ideal or imaginary concept drawn from the apparent (not real) boundaries of physical objects. We naturally associate the surface with the limited solid, not with the

surrounding air. Thus we think of the colored surface of the crystal as belonging to the crystal; and if yellow oil lies on the water in a glass, we think of the under surface of the oil as yellow and belonging only to the oil: while a mathematical surface pertains equally to the two solids that it separates.

4. These ideal mathematical surfaces may be dealt with as existing by themselves, and as movable. In illustration of this, think of a shadow.

5. A surface may be finite yet unbounded in the sense of having no abrupt or natural stopping-place on it, no visible break or obvious limit in it. Such is the surface of an egg. Set it up in an egg-cup, and run a pencil-mark around it. Then you may think of the surface of the egg as divided into three parts, the white surface within the cup, the ribbon-like black surface of the pencil-mark, and the white surface above this black ribbon.

6. Between the black surface and the two white surfaces are two boundaries which are neither black nor white. These boundaries are not thin strips of surface any more than the surface is a thin layer of solid. Where a white surface meets a black there is a common boundary of both, dividing each from the other, and belonging to both.

Fig. 2.

7. A boundary of the sort capable of wholly enclosing a piece of surface so that nothing moving in the surface could enter this piece of surface except through this boundary, but itself no surface, is called a *line*.

8. A boundary between two adjacent pieces of a line, and common to both pieces, but itself no line, is called a *point*.

Fig. 3.

9. Two lines cross or intersect in a point.

10. Two surfaces intersect in a line.

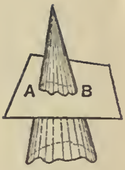


FIG. 4.

11. When a chalk mark is drawn across a blackboard, each of the two edges, neither white nor black, is a line.

12. When one chalk mark crosses another, four points are fixed by the crossing of the four edges.

FIG. 5.

13. Surfaces, lines, or points, or any combinations of them, are called figures.

14. Any figure may be looked upon as two coincident figures. Mathematical figures wholly lack impenetrability.

15. If we imagine a figure to move, we may also suppose it to leave behind its outline or *trace*.

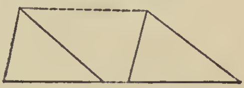


FIG. 6.

16. Two coincident figures cannot be distinguished from one another unless they be separated by moving one.

17. ASSUMPTION I. Figures may be moved about, without any other change.

18. Figures which can be made to coincide are called *congruent*.

19. If a solid has, as part of its boundary, a piece of surface which appears the same from within the solid as from without, and if any two of three such solids will fit each other all over these surfaces, then each of these surfaces is called *plane*. Such a surface unbounded is called a *plane*.

20. Any piece of a plane will slide in the plane, and after being turned over will fit the plane.

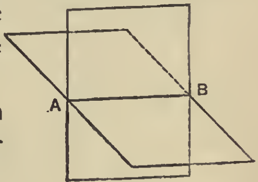


FIG. 7.

21. The intersection of two planes is called a straight line, or simply a *straight.*

22. A straight is a line in a plane which appears the same from both the regions bounded by it in the plane.

Fig. 8.

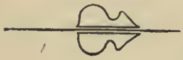


Fig. 9.

23. A piece of the plane with part of the straight as one of its boundaries would fit all along the straight from both sides.

24. Assumption II. If two straights have two points in common they coincide throughout.

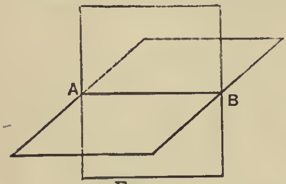


Fig. 10.

25. A straight with two points in a plane lies wholly in that plane.

For it lies in a plane, and if this is another plane the two intersect in a straight which has two points in common with the given straight.

26. *Assumed Construction* I. A straight can be drawn through any two points.

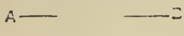


Fig. 11.

27. A *sect* is the piece of a straight between two definite points.

Fig. 12.

28. A *curve* is a line no part of which is straight.

29. Assumption III. A figure with two points fixed can still be moved, and the whole figure partakes of the motion, except the straight through the two fixed points.

Such motion is called *revolution* about this straight as axis. It may be continued until each point of the figure coincides with its trace. Such a turning is spoken of as one complete revolution, or simply, a revolution.

30. If a third point, not on the axis, be fixed, all motion of a rigid figure is prevented.

31. Three points not on a straight are necessary and sufficient to determine a plane.

32. Any straight in a plane cuts it into two parts called *hemiplanes.*

33. By a half-revolution of their plane about the common straight, either of two hemiplanes may be brought into coincidence with the trace of the other. Thus one hemiplane may be thought of as made to coincide with the other by folding over along the common axis.

34. Any point in a straight cuts it into two parts called *rays.*

Fig. 13.

35. The figure so formed is a special case of the figure formed by two rays going out from the same point, called a *bi-radial.*

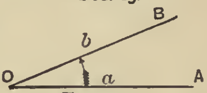


Fig. 14.

36. A bi-radial lies wholly in one plane.

37. One ray, *a*, of a bi-radial, may be brought into coincidence with the other ray, *b*, by a turning in the plane, or *rotation,* about the common point, or *vertex* O; and this turning may be in the sense indicated by the arrow in Fig. 14, or in the opposite sense.

38. To fix that sense of rotation which is to be considered as *positive* (which kind is meant if nothing else is stated), we take the turning of a ray in the sense opposite to that of the hands of a watch as positive. The watch hands, then, turn in the *negative sense.*

Clockwise is minus [—].

Counter-clockwise is plus [+].

39. A bi-radial looked at with special reference to the magnitude and sense-of-turning of a ray's rotation from one of its rays into the other, is called an *angle.*

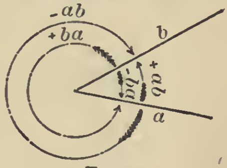


Fig. 15.

Thus, though we consider no turning beyond one complete rotation, yet the same bi-radial is four different angles,

$\measuredangle + ab$, $\measuredangle - ab$, $\measuredangle + ba$, $\measuredangle - ba$, where the turning is always from the first-mentioned ray into the second.

40. If O (Fig. 14) is the origin or initial point of ray a and of ray b, and A any other point on a, and B on b, then $\measuredangle + ab$ may be written $\measuredangle + OA/OB$, or even $\measuredangle + AOB$, where the order of the letters denotes that the angle is generated by a ray rotating about O from OA to OB, and the sign fixes the sense of that rotation.

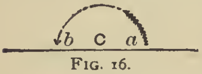


Fig. 16.

41. If a ray, a, is turned about the initial point, C, until it coincides with the continuation, b, of its trace beyond C, the angle ab is called a *straight angle*.

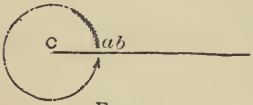


Fig. 17.

42. If we turn still more, until the moving ray has made a complete rotation, and coincides with its trace, the angle is called a *perigon*.

43. If $\measuredangle ab$ equals a perigon, then the ray a coincides with the ray b.

44. When a bi-radial is looked upon as an angle, its two rays are called the *arms* of the angle.

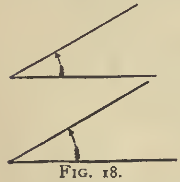


Fig. 18.

45. Two angles are *equal* if they can be so placed that their arms and therefore their vertices coincide, and that both are described simultaneously by the turning of the same ray about their common vertex.

46. Equality implies that both angles have the same sense.

47. Two angles which can be made equal by changing the sign of one, are said to be equal in magnitude but opposite in sense.


Fig. 19.

48. Since turning the plane of a bi-radial through half a revolution changes the sense of each of its four angles, therefore, if one angle by folding over along an axis is made equal to

another, then the angles were equal in magnitude but opposite in sense.

49. ASSUMPTION IV. All straight angles are equal in magnitude.

50. As a consequence, all perigons are equal in magnitude.

51. If two angles have a vertex and an arm in common, they are called *adjacent angles*.

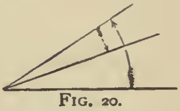


FIG. 20.

52. When two adjacent angles are of the same sense, and so situated that they cannot be simultaneously described, even

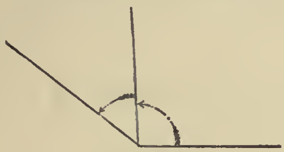


FIG. 21.

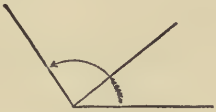


FIG. 22.

in part, by the same ray rotating, their *sum* is the angle of like sense whose arms are their two non-coincident arms.

53. When the sum of any two angles is a straight angle, each is said to be the *supplement* of the other.


FIG. 23.

54. When the sum of any two angles is a perigon, each is said to be the *explement* of the other.

Thus $\measuredangle + ab$ and $\measuredangle + ba$ are explemental.

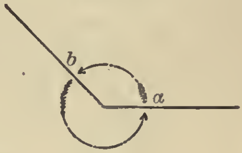
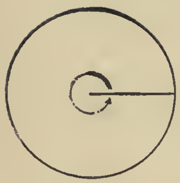


FIG. 24.

CHAPTER II.

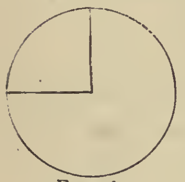
THE CIRCLE.



55. If, in a plane, a sect turns about one of its end points the other end point describes a curve called a *circle*.

FIG. 25.

56. The fixed end point is called the *center* of the circle.

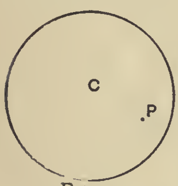


57. Any sect from the center to a point on the curve is called a *radius*.

58. All radii are equal, being equal to the generating sect.

FIG. 26.

59. Since the moving sect, after rotating through a perigon, returns to its trace, therefore the moving end point describes a closed curve.



60. This curve divides the plane into two parts, one of which is finite and is swept over by the moving sect.

61. This finite plane surface is called the *surface of the circle*. Any point in this finite plane is said to lie within the circle.

FIG. 27.

62. *Assumed Construction* II. A circle may be described from any given point as center with any given sect as radius.

63. A *theorem* is a statement usually capable of being inferred from other statements previously accepted as true.

64. A *corollary* to a theorem is a statement whose truth follows readily from that of the theorem.

65. A theorem consists of two parts, the *hypothesis* (that which is assumed), and the *conclusion* (that which is asserted to follow therefrom).

66. A *problem* is a proposition in which something is required to be done by a process of construction.

67. The treatment of a problem in elementary geometry consists,—

[1] *Construction.* In indicating how the ruler and compasses are to be used in effecting what is required.

[2] *Proof.* In showing that the construction so given is correct.

[3] *Determination.* In fixing whether there is only a single solution, or suitable result of the indicated construction; or more than one; and in discussing the limitations, which sometimes exist, within which alone the solution is possible.

68. Our assumed constructions allow the use of the straight-edge not marked with divisions, for drawing and producing sects, and the use of compasses for drawing circles and the transference of sects. It is important to note the implied restriction, namely, that we work in the plane, and that no construction in elementary geometry is allowable which cannot be effected by combinations of these two primary constructions.

69. Theorem. The sect to a point, from the center of a circle, is less than, equal to, or greater than the radius, according as the point is within, on, or without the circle.

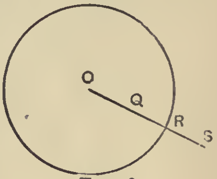


FIG. 28.

Proof. For any point Q, within the circle, lies on some radius, OQR. If S is without the circle, then the sect OS contains the radius OR.

70. Inverse. A point is within, on, or without the circle,

according as its sect from the center is less than, equal to, or greater than the radius.

71. Theorem. Circles of equal radii are congruent.

Proof. For, if put in the same plane, with centers in coincidence, every point of each is on the other, because of the equality of their radii. Thus $\odot C\,[r] \cong \odot O\,[r]$.

72. Corollary. A circle turned about its center slides on its trace.

This fundamental property of this curve enables us to turn any figure connected with the circle about the center without changing the relation to the circle.

73. Circles which have its same center are called *concentric*.

74. Concentric circles with a point in common coincide.

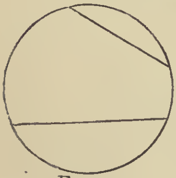


Fig. 29.

75. A sect whose end points are on the circle is called a *chord*.

76. Any chord through the center is called a *diameter*.

77. All diameters are equal, each being equal to two radii.

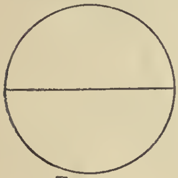


Fig. 30.

78. Every diameter is bisected by the center of the circle.

79. No circle can have more than one center.

For, if it had two, the diameter through them would have two mid points.

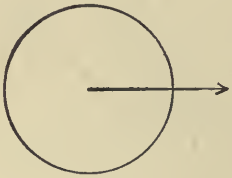


Fig. 31.

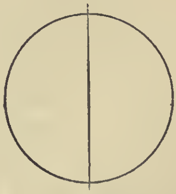


Fig. 32.

80. Any ray from the center of a circle cuts the circle in one, and only one, point.

81. Any straight through its center cuts the circle in two and only two points.

82. Any piece of a circle is called an *arc*.

83. When the end point of a radius describes an arc, the radius rotates through an angle having its vertex at the center. This angle is called the *angle at the center*, and is said to be *subtended by* the arc simultaneously described, or to *stand upon* that arc.

Fig. 33.

84. An arc, being described by the end point of a rotating radius, is said to have the same sense as the angle through which that radius rotates.

85. Arcs congruent and of the same sense are called equal.

86. The sum of two arcs, of the same circle, or of equal circles, is the arc which subtends an angle at the center equal to the sum of the angles subtended by those arcs separately.

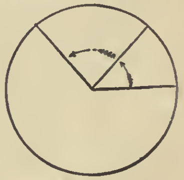


Fig. 34.

87. Theorem. Equal arcs subtend equal angles at the center, and, inversely, equal angles at the center stand upon equal arcs.

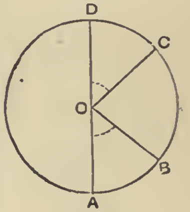


Fig. 35.

Proof. For, if arc AB equal arc CD, we may slide the arc AB, together with the radii OA and OB, along the circle until A coincides with C; then will B coincide with D, since arc CD equals arc AB.

Therefore $\measuredangle$ AOB will coincide with $\measuredangle$ COD, and will be equal to it in magnitude and sense.

88. It follows, that if A, B, C, etc., denote points on the circle and a, b, c, etc., the radii drawn to those points, then every equation between arcs AB, BC, etc., will carry with it an equation between the corresponding angles ab, bc, etc.; and inversely.

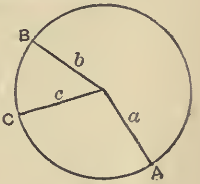


Fig. 36.

89. Theorem. In the same or equal circles, of two unequal arcs, the greater subtends the greater angle at the center.

Fig. 37.

Proof. If the first arc is greater than the second, it equals the second plus a third arc, and so the angle which the first subtends is greater than the angle which the second subtends by the angle which the third arc subtends at the center.

90. Inversely: Of two unequal angles at the center, the greater intercepts the greater arc.

91. Two arcs which together equal the whole circle are called *explemental.*

Thus the explemental angles at the center of a circle, whose arms are the same radii, are said to stand upon the explemental arcs which would be described simultaneously with the angles, the greater angle upon the greater arc.

92. Explemental arcs equal in magnitude are called *semicircles.*

93. A semicircle subtends a straight angle. For two subtend a perigon, and are equal.

94. Any straight through the center cuts the circle into two semicircles. For it makes at the center straight angles which together are subtended by the whole circle.

95. If we fold over about a straight through the center of a circle, the semicircles it makes are brought into coincidence.

For every point on the turned semicircle must fall on some point of the other, as its sect from the center is a radius.

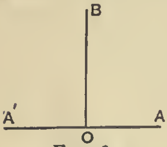


Fig. 38.

96. Two arcs which together equal a semicircle are called *supplemental.*

97. Half a straight angle is called a *right angle.*

98. All right angles are equal in magnitude.

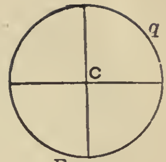


FIG. 39.

99. The arc subtending a right angle is called a *quadrant*. It is one quarter of a circle.

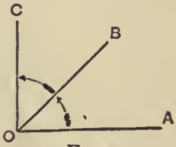


FIG. 40.

100. Two straights which make a right angle are said to be *perpendicular* to one another.

101. Two angles whose sum is a right angle are called *complemental.*

102. An angle less than a right angle is called *acute.*

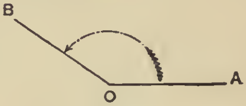


FIG. 41.

103. An angle greater than a right angle, but less than a straight angle, is called *obtuse.*

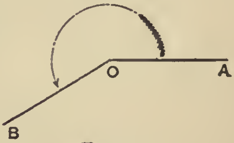


FIG. 42.

104. An angle greater than a straight angle, but less than a perigon, is called *reflex*.

105. An angle which is either acute, right, or obtuse, is called a *minor angle.*

106. An arc less than a semicircle is called a *minor arc.*

107. An arc less than a circle, but greater than a semicircle, is called a *major arc.*

108. Theorem. If two circles have one common point not on the straight through their centers, they have also another such point.

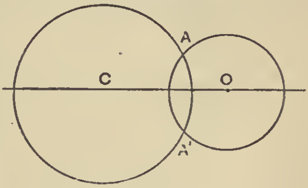


FIG. 43.

Proof. Let ⊙ C and ⊙ O have the the point A in common. Fold the figure over along the straight through their centers, CO. Then the semicircles which have A in common are brought into coincidence with the other semicircles. Therefore these also have a common point, A'.

109. Theorem. If two circles have a common point not on the straight through their centers, and therefore another such point, then the center-straight bisects the angles made at the centers by the radii to these two common points, and is the perpendicular bisector of the common chord.

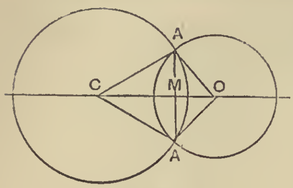


FIG. 44.

Proof. For by folding over along CO we bring A into coincidence with A'. Therefore sect $AM =$ sect $A'M$. $\measuredangle\, OMA = \measuredangle\, A'MO$. $\measuredangle\, MCA = \measuredangle\, A'CM$. $\measuredangle\, AOM = \measuredangle\, MOA'$.

CHAPTER III.

THE FUNDAMENTAL PROBLEMS.

110. Problem. To bisect any given sect.

Construction. With its end points, A and A', as centers, and itself as radius, describe two circles. They will have one common point not on their center straight, and therefore a second such. Join these two common points, C and O. Then CO bisects the given sect AA'.

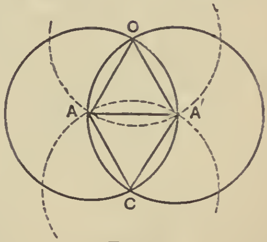


FIG. 45.

Proof. For AA' is $\perp$ to CO, and if from C and O as centers, with radii equal to AA', two circles were described, then AA' would be a common chord, bisected by the center-straight CO.

111. Theorem. The straight through the mid point of a chord, and the center of the circle, is perpendicular to the chord, and bisects the explemental arcs, and their angles at the center.

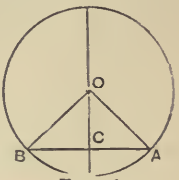


FIG. 46.

Proof. A and B are any points on $\odot O$. Turn the whole figure over and apply it to its trace, so that O falls on O, but A on the trace of B, and B on the trace of A. Then the bisection point, C, of the chord AB falls on its own trace, and consequently the whole change amounts only to half a revolution of the figure about the straight CO.

112. Problem. To bisect any given angle.

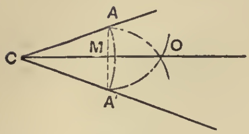


Fig. 47.

Construction. With its vertex, C, as center, and any sect, r, as radius, describe a circle cutting the arms of the angle at A and A'. Bisect the chord AA', and join its mid point, M, to the center C. Then MC bisects $\measuredangle ACA'$.

113. Problem. At a given point on a given straight, to draw a perpendicular to that straight.

Construction. Bisect the straight angle at the point.

114. Problem. Through a given point, not in a given straight, to draw a perpendicular to that straight.

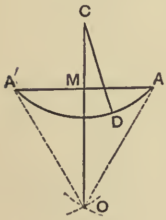


Fig. 48.

Construction. In the hemiplane not containing the given point, C, take any point D. Call A and A' points where $\odot C[CD]$ cuts the given straight. Bisect the chord AA' at M. Then is $CM \perp$ to AA'.

Determination. Through a given point only one perpendicular can be drawn to a given straight. For, if the plane were folded over along the given straight, the given point would fall on the production of any perpendicular from it to the straight.

115. Since the perpendicular from the center to a chord of a circle bisects that chord, and also the explemental arcs and the explemental angles pertaining to that chord, therefore the r't bi' of any chord passes through the center; and the straight which possesses any two of these seven properties possesses also the other five.

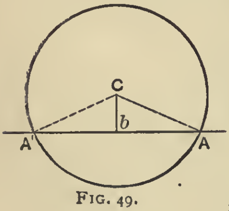


Fig. 49.

116. Problem. To bisect any given arc.

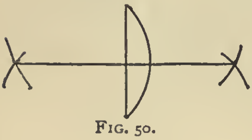


Fig. 50.

Construction. Join its extremities, and draw the r't bi' of this chord.

117. Theorem. A straight cannot have more than two points in common with a circle.

Proof. For, if it had a third, then, since the r't bi' of a chord contains the center, there would be three perpendiculars from the center to the same straight.

118. Theorem. Every point which joined to two points gives equal sects is on the perpendicular bisector of the sect joining those two points.

Proof. The r't bi' of the chord contains the center.

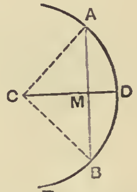


FIG. 51.

119. Corollary. Circles with three points in common coincide.

CHAPTER IV.

SYMMETRY AND SYMCENTRY.

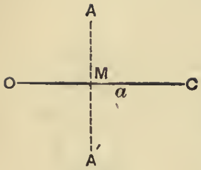


FIG. 52.

120. Two points are said to be *symmetrical* with regard to a given straight, called the Axis of Symmetry, when the axis bisects at right angles the sect joining the two points.

121. Two points have always one, and only one, symmetry axis.

122. A point has, with regard to a given axis of symmetry, always one, and only one, symmetrical point; namely, the one on the ray from the given point perpendicular to the axis, which ends the sect bisected by the axis.

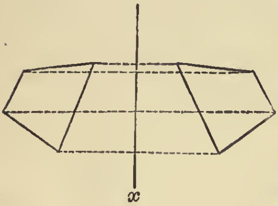


FIG. 53.

123. Two figures have an axis of symmetry when, with regard to this straight, every point of each has its symmetrical point on the other.

124. Two figures are symmetrical when they can be placed so as to have an axis of symmetry.

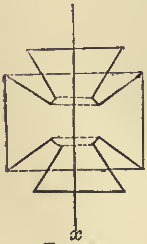


FIG. 54.

125. One figure has an axis of symmetry when, with regard to this straight, every point of the figure has its symmetrical point on the figure.

126. One figure is symmetrical when it has an axis of symmetry.

127. Any figure has, with regard to any given straight as axis, always one, and only one, symmetrical figure.

128. One figure is symmetrical when it has an axis with regard to which its symmetrical figure coincides with itself.

129. Every point in the axis is symmetrical to itself.

130. The axis is symmetrical with regard to itself.

131. Two points are said to be *symcentral* with regard to the mid point of their joining sect.

FIG. 55.

132. A point has, with regard to a given symcenter, always one, and only one, symcentral point; namely, the one on the ray from the given point through the symcenter, which ends the sect bisected by the symcenter.

133. Two figures have a symcenter when, with regard to this point, every point of each has its symcentral point on the other.

134. Two figures are symcentral when they can be placed so as to have a symcenter.

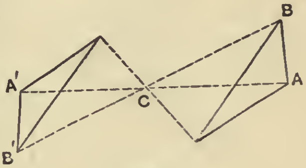


FIG. 56.

135. One figure has a symcenter when, with regard to this point, every point on the figure has its symcentral point on the figure.

136. One figure is symcentral when it has a symcenter.

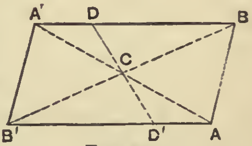


FIG. 57.

137. Any figure has with regard to any given point as symcenter, always one, and only one, symcentral figure.

138. One figure is symcentral when it has a point with regard to which its symcentral figure coincides with itself.

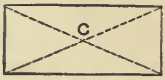


FIG. 58.

139. Theorem. A straight, or sect, or angle, in one of two symmetrical [or symcentral] figures, has a symmetrical [or symcentral] straight, or sect, or angle, in the other.

Proof. For by half a revolu-

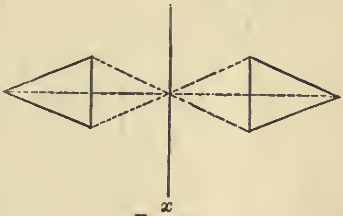


FIG. 59.

tion [rotation] of one figure about the axis [symcenter] the two are made to coincide.

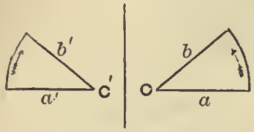


FIG. 60.

140. If point $C \cdot|\cdot C'$, and from C rays a and b $\cdot|\cdot$ rays a' and b' from C', the $\measuredangle + ab \cdot|\cdot \measuredangle - a'b'$, remembering that in any angle the turning is from the first-mentioned ray into the second, and the sign denotes the sense of that turning.

141. The intersection point of two straights is symmetrical [or symcentral] to the intersection of two symmetrical [or symcentral] to those.

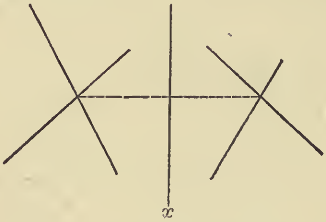


FIG. 61.

FIG. 62.

142. The intersection point of two symmetrical straights is on the axis.

143. If three points lie in a straight, their symmetrical [or symcentral] points lie in a symmetrical [or symcentral] straight.

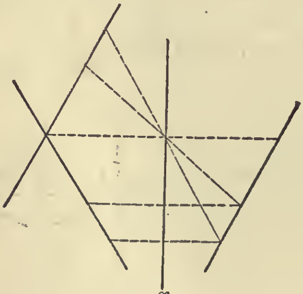


FIG. 63.

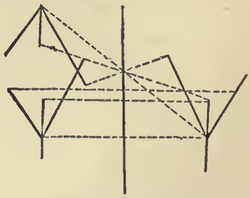


FIG. 64.

144. The bisector of an angle is symmetrical [or symcentral] to the bisector of the symmetrical [or symcentral] angle.

145. Every point symmetrical [or symcentral] to itself lies in the axis [or symcenter].

146. The angle between two symmetrical straights is bisected by the axis.

FIG. 65.

147. Any straight is symmetrical with regard to any of its perpendiculars.

148. Any straight is symcentral with regard to any of its points.


FIG. 66.

Thus the intersection point of two straights is a symcenter for each; so the non-adjacent or *vertical* angles are equal, and their bisectors, being symcentral rays from the symcenter, are in one straight.

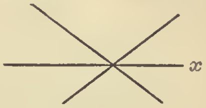


FIG. 67.

149. Theorem. Two intersecting straights are symmetrical with regard to either of their angle bisectors.

Proof. For the points which would be brought into coincidence by folding along this bisector were symmetrical with regard to it.

150. Any circle is symmetrical with regard to any of its diameters, and symcentral with regard to its center.

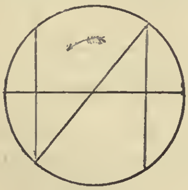


FIG. 69.

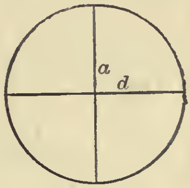


FIG. 70.

151. Every axis of symmetry of a circle passes through the center.

For the diameter perpendicular to this axis is bisected by it.

152. A figure made up of a straight and a point without it is symmetrical, but never symcentral.

CHAPTER V.

TANGENTS.

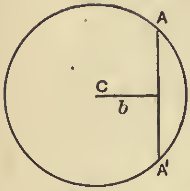


FIG. 71.

153. Theorem. Every point on the perpendicular bisector of a sect is the center of a circle passing through its end points.

For $A \cdot\!|\!\cdot A'$, axis b; $\therefore$ $CA = CA'$.

Thus sects from any point on its perpendicular bisector to the end points of the sect are equal.

154. A straight which has two points in common with a circle is called a *secant*.

155. A straight which has only one point in common with a circle is called a *tangent* to the circle, and the point is called the *point of contact*.

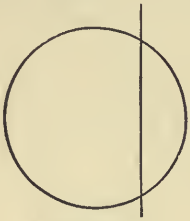


FIG. 72.

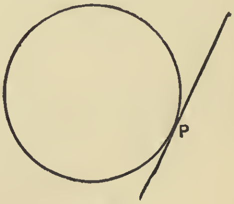


FIG. 73.

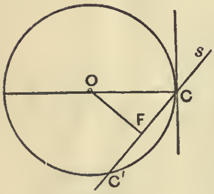


FIG. 74.

156. Since any chord is bisected by the perpendicular from the center, $\therefore$ a straight $\perp$ to a diameter at an end point has only this point in common with the circle.

This point of the circle is symmetrical to itself with regard to this diameter as axis. But if we draw through this point C any

straight s, not $\perp$ to the radius, then the perpendicular from the center, O, will meet this straight s at some other point, F. Hence the straight s cuts the circle again at $C' \cdot|\cdot C$, axis OF. Therefore:

Theorem. At every point on the circle one, and only one, tangent can be drawn, namely, the perpendicular to the radius at the point.

157. The perpendicular to a tangent from the center of a circle cuts it in the point of contact.

158. The perpendicular to the tangent at the point of contact contains the center.

159. The radius to the point of contact of a tangent is perpendicular to the tangent.

160. To draw the tangent to a circle at any point, draw the perpendicular to the radius at that point.

161. Let O be a point not in the straight s, and $OC \perp$ to s: then s is tangent to $\odot O\ [OC]$ at C.

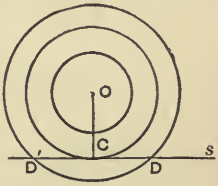


FIG. 75.

Any second circle concentric with the first, but of lesser radius, lies wholly within the first.

A third concentric circle, with radius $> OC$, lies wholly without the $\odot O\ [OC]$, and cuts s in $D \cdot|\cdot D'$, axis OC; $\therefore CD = CD'$.

A fourth concentric circle, with radius $> OD$, lies wholly without the third; $\therefore$ its intersections with s lie without the sect DD'.

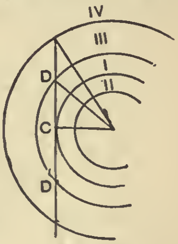


FIG. 76.

Hence the four following theorems:

162. A straight will be a secant, a tangent, or not meet the circle, according as the perpendicular to it from the center is less than, equal to, or greater than the radius.

163. The perpendicular is the least sect between a given point and a given straight.

164. Except the perpendicular, any sect from a point to a straight is called an *oblique*.

165. Two obliques from a point to a straight, making equal sects from the foot of the perpendicular, are equal.

166. Of any two obliques between a point and a straight, that which makes the greater sect from the foot of the perpendicular is the greater.

167. Problem. From a given point without the circle to draw a tangent to the circle.

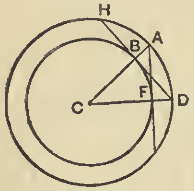


FIG. 77.

Construction. Join the given point A to the center C, cutting the circle in B. Draw $BD \perp$ to CB, and cutting in D the $\odot C\,[CA]$. Join DC, cutting $\odot C\,[CB]$ in F. Then AF is tangent to $\odot C\,[CB]$.

Proof. Radius CA, $\perp$ to chord HD, bisects arc HD; $\therefore$ if we rotate the figure until H comes upon the trace of A, then A is on the trace of D, $\therefore$ tangent HB on trace of AF.

Determination. Always two and only two tangents.

168. Corollary. By symmetry the straight through C, the center of a circle, and A an external point, bisects the angle between the two tangents from A to $\odot C$, and also the angle between the radii to the points of contact, F and F'; and sect $AF =$ sect AF'.

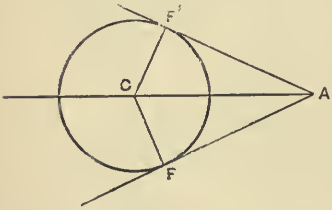


FIG. 78.

169. Corollary. Any point from which the perpendiculars on two intersecting straights are equal, is on one of their angle bisectors. It is center of a circle to which they are tangent.

170. Corollary. The centers of all circles tangent to two intersecting straights are in their angle bisectors.

171. Inversely. From any point on a bisector of an angle

made by two straights, the perpendiculars to those straights are equal.

For the bisector is a symmetry axis for the two straights: so when we fold along it, the foot of the perpendicular to one straight falls on the other straight, and there is only one perpendicular from a point to a straight.

CHAPTER VI.

CHORDS.

172. Take AB any chord in $\odot O$. The $\odot A$ $[AB]$ cuts $\odot O$ in two points, $B \cdot|\cdot B'$, axis AO [the center-straight]. But the end points of all sects from A which are equal to AB must lie on $\odot A$ $[AB]$; whence:

Theorem. Chords from any point of a circle are equal in pairs, one on each side of the diameter from that point.

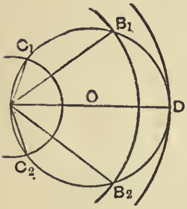


Fig. 79.

173. A circle $= \odot O$, and containing a chord $= AB$, can be superimposed upon $\odot O$, and then rotated until one end of the chord comes at A. The other end of this chord then lies on both $\odot O$ and $\odot A$ $[AB]$, and so falls on B or B'; and the chord coincides with AB or AB'. Hence the theorem:

In the same or equal circles, to equal chords pertain equal minor arcs.

174. Corollary. In the same or equal circles of arcs pertaining to equal chords any two are either equal or explemental.

175. If with center A and radius $AC < AB$ we describe a second circle, it will lie wholly within $\odot A$ $[AB]$. Consequently it cuts $\odot O$ in points C and C' on the arc BAB'; $\therefore$ arc $AC <$ arc AB. Thus if the chord decreases, so does the minor arc; and inversely, of two unequal minor arcs, the greater has the greater chord.

176. If the chord increases, its major arc decreases, since its major and minor arcs are always explemental. Inversely, if a major arc decreases, its chord increases.

177. A diameter is the greatest chord. Every other chord equals a piece of the diameter.

178. Equal arcs, being congruent, have equal chords.

Therefore, also, explemental arcs have equal chords.

179. Equal chords, having equal minor arcs, which may be brought into coincidence by rotation about the center, have also equal perpendiculars from the center.

180. In the same or equal circles, chords which have equal perpendiculars from the center, since by rotation one may be put upon the other, are equal.

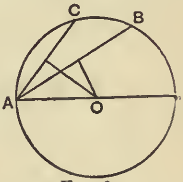


FIG. 80.

181. Since the end point C of a chord $AC < AB$ lies on the minor arc AB, $\therefore$ it is on the side of AB remote from the center O; $\therefore$ the mid-point of chord AC is on the side of AB remote from O.

Theorem. In the same or equal circles, the greater chord has the lesser perpendicular from the center.

182. Inversely. The chord with the greater perpendicular from the center is the lesser; for [181] it cannot be the greater chord, nor [179] can they be equal.

183. Problem. At a given point G, in a given straight s, to make an angle equal to a given angle ACB.

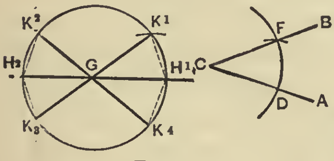


FIG. 81.

Construction. With any radius draw $\odot C\,[r]$, cutting CA at D, and CB at F. Join DF. Draw $\odot G\,[r]$, cutting the given straight at H. Draw $\odot H\,[r' = DF]$, cutting $\odot G\,[r]$ at K. Join GK. Then $\measuredangle HGK = \measuredangle ACB$.

Proof. $\odot G\,[r] = \odot A\,[r]$, and chord $HK =$ chord DF.

$\therefore$ minor arc $HK =$ minor arc DF.

$\therefore$ minor $\measuredangle HGK =$ minor $\measuredangle DCF$.

Determination. The construction will give four minor angles at G, all equal, namely,

$$H_1GK_1;\ K_2GH_2;\ K_4GH_1;\ H_2GK_3.$$

CHAPTER VII.

TWO CIRCLES.

184. A figure formed by two circles is symmetrical with regard to their center-straight as axis.

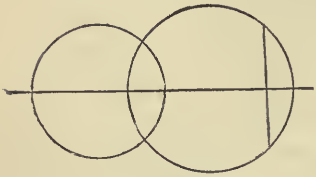


Fig. 82.

Every chord perpendicular to this axis is bisected by it.

If the circles have a common point on this straight, they cannot have any other point in common, for any point in each has its symmetrical point with regard to this axis, and circles with three points in common coincide.

185. Two circles with only one point in common are called tangent, are said to touch; and the common point is called the point of tangency or contact.

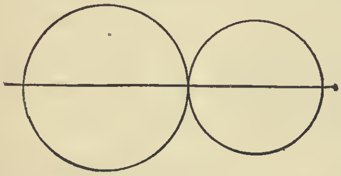


Fig. 83.

186. If two circles touch, then, since there is only one common point, this point of contact lies on the center-straight, and a perpendicular to the center-straight through the point of contact is a common tangent to the two circles.

CHAPTER VIII.

PARALLELS.

187. A straight cutting across other straights is called a *transversal.*

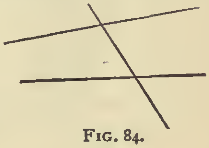


FIG. 84.

[In plane geometry, all are in one plane.]

188. If, in a plane, two straights are cut in two distinct points by a transversal, at each of these points four positive minor angles are made.

Of these eight angles, four are between the two straights [namely, 3, 4, *a*, *b*], and are called Interior Angles: the other four lie outside the two straights, and are called Exterior Angles.

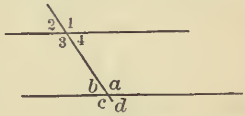


FIG. 85.

Angles, one at each point, which lie on the same side of the transversal, the one exterior and the other interior, are called Corresponding Angles [e.g., 1 and *a*].

Two non adjacent angles on opposite sides of the transversal, and both interior or both exterior, are called Alternate Angles [e.g., 3 and *a*].

Two angles on the same side of the transversal, and both interior or both exterior, are called Conjugate Angles [e.g., 4 and *a*].

189. Theorem. If two corresponding or two alternate angles are equal, or two conjugate angles are supplemental, then every angle is equal to its corresponding and to its alternate, and supplemental to its conjugate.

[Use vertical angles and supplemental adjacent angles.]

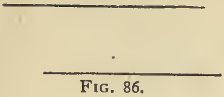


Fig. 86.

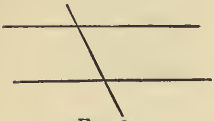


Fig. 87.

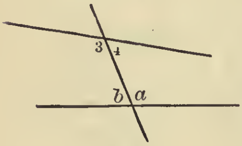


Fig. 88.

190. *Parallels* are straights in the same plane which nowhere meet.

[Note. As we are working on a plane, the clause "in the same plane" would be understood even if not mentioned.]

191. Assumption V. Two coplanar straights are parallel if a transversal makes equal alternate angles.

192. Assumption VI. If two coplanar straights cut by a transversal have a pair of alternate interior angles unequal, they meet on that side of the transversal where lies the smaller angle.

193. Theorem. If two straights cut by a transversal have corresponding angles equal, or conjugate angles supplemental, they are parallel.

For either hypothesis makes the alternate angles equal.

194. If two straights cut by a transversal have conjugate angles not supplemental, they meet.

For the alternate angles are unequal.

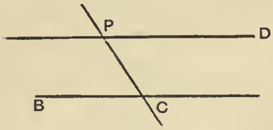


Fig. 89.

195. Problem. Through a given point to draw a parallel to a given straight.

Construction. Join the given point P to any point C, of the given straight CB. Then at P make an angle CPD alternate and equal to $\measuredangle PCB$.

Determination. There is only one solution.

196. Corollary. Two coplanar straights parallel to the same straight are parallel to one another.

For they cannot meet.

197. Theorem. If a transversal cuts two parallels, the alternate angles are equal.

Proof. For if they were unequal, the straights would meet.

198. Theorem. Any two parallels are symcentral with regard to the mid point of the sect which they intercept on any transversal.

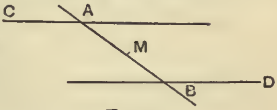


FIG. 90.

Proof. Rotating the figure about M through a straight angle brings A into coincidence with the trace of B and ∡ CAM into coincidence with the trace of the equal alternate ∡ DBM.

199. Two angles with their arms parallel are either equal or supplemental [189 and 197.]


FIG. 91.

200. If two angles have their arms respectively perpendicular, they are either equal or supplemental.

For rotating one of the angles through a r't ∡ around its vertex, its arms become ⊥ to their traces, and ∴ ∥ to the arms of the other ∡.

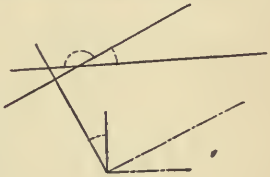


FIG. 92.

201. Points all in the same straight are called *costraight*.

202. Problem. To pass a circle through any three points not costraight.

Construction. Join the three points by three sects; to these sects erect r't bisectors; of these every two will meet, since they make an angle = or supplemental to that to whose arms they are ⊥. Suppose two to meet at C. This point joined to the three points gives three equal sects.

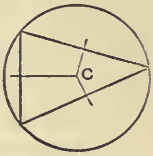


FIG. 93.

Therefore it is the center of a circle containing the three given points.

203. Corollary. The center of any ⊙ through the three points must lie on all three r't bi's.

∴ the third r't bi' passes through O.

204. Problem. To describe a circle touching three given intersecting straights not all through the same point.

Construction. At each of two intersection points draw the two angle-bisectors. Every pair of these meet, since they make conjugate angles which are not supplemental. [Two of

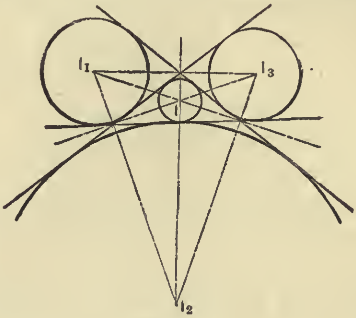


FIG. 94.

the four different angles bisected are together less than a straight angle; the other two each less than a straight angle, and the angle between bisectors of supplemental adjacent angles is right.]

From any point, as I, on a bisector through A and one through B, drop a perpendicular upon one of the given straights, as AB. A circle described with this perpendicular as radius is tangent to AB; but it also touches the second given straight BC [I lies on the bisector of an angle between AB and BC], and the third CA [I is on a bisector of an angle between AB and AC].

Determination. Every intersection point of two angle-bisectors has thus equal perpendiculars to the three given straights. It is therefore on a third angle-bisector.

Thus the four intersection points of the two bisectors through A with the two through B are the eight intersection points of the two bisectors through C with the other four.

Thus the two bisectors through the third point give no new intersections, and there are just four solutions.

205. Problem. To draw a common tangent to two given circles.

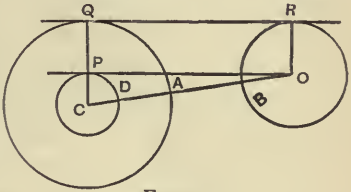


FIG. 95.

Construction. A and B are the points where $\odot C\,[CA]$ and $\odot O\,[OB]$ are cut by CO. Suppose $CA > OB$. From AC or AO cut off $AD = OB$. Describe $\odot C\,[CD]$. To it, from O, draw tangent OP. Let CP cut $\odot C\,[CA]$ in Q. Through O, on the same side of OP as Q, draw $OR \parallel$ to CP, cutting $\odot O\,[OB]$ in R. Then QR is a common tangent.

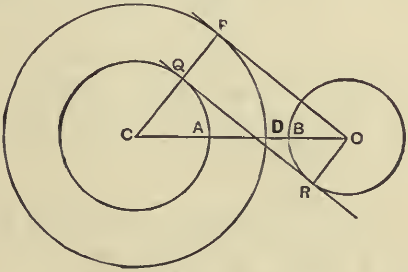


FIG. 96.

Proof. Radii $CQ = CA$, $CP = CD$; $\therefore$ $PQ = AD = OB = OR$.

But $OR \parallel$ to PQ and $\measuredangle\ OPQ$ a r't $\measuredangle$; $\therefore$ $\measuredangle\ POR$ is a r't $\measuredangle$.

$\therefore$ Q ⫞ R, axis $\perp$ to OP, $\therefore$ $OP \parallel$ to QR, $\therefore$ $\measuredangle\ PQR = \measuredangle\ QRO =$ a r't $\measuredangle$.

CHAPTER IX.

THE TRIANGLE.

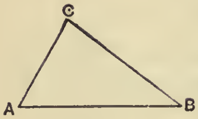


Fig. 97.

206. Three points A, B, C, not co-straight, and the three straights they determine, form a figure called a *triangle*.

207. The three points of intersection are the three *vertices* of the triangle [A, B, C, of $\triangle ABC$].

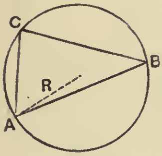


Fig. 98.

208. The circle through the vertices of a triangle is called its *circumcircle*, $\odot\, O\ [R]$, and the center O of the circumcircle is called the *circumcenter* of the triangle; its radius, R, the *circumradius*.

209. The three sects joining the vertices are the *sides* of the triangle. The side opposite the angle A is called a; opposite $\measuredangle B$ is side b; opposite C, c.

210. Straights which all intersect in the same point are called *concurrent*.

211. The three perpendicular bisectors of the sides of a triangle are concurrent in its circumcenter.

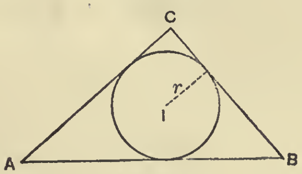


Fig. 99.

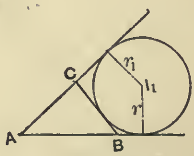


Fig. 100.

212. The circle tangent to the three sides of a triangle is called its *in-circle*, $\odot I$ [r], and its center I, the triangle's in-center [r, *in-radius*].

213. The three internal bisectors of the angles of a triangle are concurrent in its in-center.

214. A circle touching one side of a triangle and the other two sides produced is called an *escribed* circle, or *ex*-$\odot$.

The three centers I_1, I_2, I_3 of the escribed circles $\odot I_1$ [r_1], $\odot I_2$ [r_2], $\odot I_3$ [r_3] of a triangle are called its *ex-centers*.

215. The sum of two sects is the sect obtained by placing them on the same straight, with one end point of each in coincidence, but no other point in common.

216. An *exterior angle* of a triangle is one between a side and the continuation of another side.

217. Through the vertex B of a $\triangle$ draw $BD \parallel$ to AC. The exterior $\measuredangle$ ABE is made up of $\measuredangle ABD = \measuredangle BAC$ [alternate], and $\measuredangle DBE = \measuredangle ACB$ [corresponding]. Therefore:

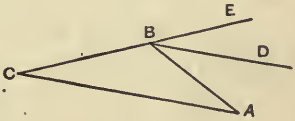


FIG. 101.

Theorem. In every triangle any exterior angle equals the sum of the two interior angles not adjacent to it. Therefore:

218. Theorem. The sum of the angles in any plane triangle is a straight angle.

219. Corollary. In a triangle, at least two angles are acute. The third angle may be acute, right, or obtuse; and the triangle is called acute-angled, right-angled, or obtuse-angled, accordingly.

220. In a right-angled triangle the side opposite the right angle is called the *hypothenuse*.

221. A triangle with two sides equal is called *isosceles*.

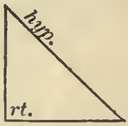


FIG. 102.

222. Theorem. If one side of a triangle be greater than a second, the angle opposite the first must be greater than the angle opposite the second.

Proof. Given $BA > BC$. Draw bisector BD of $\measuredangle B$, and fold over along this axis. Then C falls on BA at C' between B and A. Then $\measuredangle C$ now appears as an exterior $\measuredangle$ to $\triangle AC'D$, and $\therefore > \measuredangle A$ not adjacent.

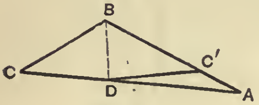


FIG. 103.

223. Theorem. If one angle of a triangle is greater than a second, the side opposite the first must be greater than the side opposite the second.

Proof. Given $\measuredangle C > \measuredangle A$. Draw the bisector BD of $\measuredangle B$. Then is $\measuredangle ADB$ $[= \measuredangle (C + \frac{1}{2}B)]$, $> \measuredangle BDC$ $[= \measuredangle (A + \frac{1}{2}B)]$; therefore on folding over along the axis BD, $\measuredangle BDC$ will fall within $\measuredangle ADB$, and therefore C must fall between A and B.

224. Corollary I. In an isosceles triangle, the angles opposite the equal sides are equal.

225. Corollary II. If two angles of a triangle are equal, the triangle is isosceles.

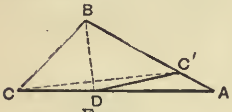


FIG. 104.

226. If we join CC' in the preceding figure then $\measuredangle DCC' = \measuredangle CC'D$, since $\triangle DC'C$ is isosceles; $\therefore \measuredangle ACC' < \measuredangle CC'A$; $\therefore AC' < AC$. But $AC' = AB - BC$. Therefore:

Theorem. In every triangle the difference of two sides is less than the third side.

$AB - BC < AC$; $\therefore AB < AC + BC$. Therefore:

227. Theorem. In every triangle the sum of any two sides is greater than the third side.

CHAPTER X.

POLYGONS.

228. A number of sects, the second beginning at the end point of the first, the third at the end point of the second, etc., are called a *broken line*.

FIG. 105.

229. Theorem. The sect between two given points is smaller than any broken line between the points.

Proof. Beginning at one of the points, reduce the number of sects in the broken line, and its size, by substituting for the first two the sect joining their non-coincident end-points. So proceed until the sect between the two given points is attained.

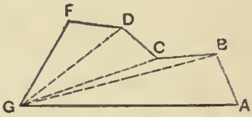


FIG. 106.

230. If, in a broken line, the beginning point of the first sect coincides with the ending point of the last, the figure is called a *polygon*, the broken line its *perimeter*, and the sects its sides.

231. A polygon has as many angles as it has sides.

232. A polygon, no side of which cuts another, is called an *undivided polygon*.

233. In a plane, the perimeter of an undivided polygon encloses one finite uncut piece, which is called the *surface of the polygon*.

234. By the angles of an undivided polygon we understand those each described by a ray sweeping over part of the surface of the polygon.

235. An undivided polygon each of whose angles is less than a straight angle is called *convex.*

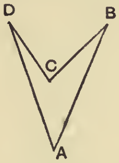


FIG. 107.

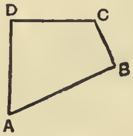


FIG. 108.

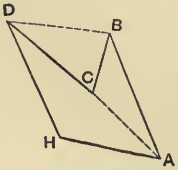


FIG. 109.

236. Any sect joining vertices not consecutive is called a *diagonal* of the polygon.

237. A polygon of three angles is a trigon or triangle; one of four angles is a tetragon; of five, a pentagon; of six, a hexagon; of seven, a heptagon; of eight, an octagon; of nine, a nonagon; of ten, a decagon; of twelve, a dodecagon; of fifteen, a quindecagon.

238. By the word *quadrilateral,* unqualified, we will mean an undivided tetragon.

239. A polygon both equilateral and equiangular is called *regular.*

FIG. 110.

FIG. 111.

240. A regular polygon whose sides intersect is called a *star polygon.*

241. A regular polygon, if undivided, is convex.

FIG. 112.

242. Theorem. In a plane, the sum of the angles of an undivided polygon is two less straight angles than it has sides.

Proof. By a diagonal within the polygon cut off a triangle. This diminishes the number of sides by one,

and the sum of the angles by a straight angle. So reduce the sides to three. We have left two more sides than straight angles.

243. If through its second end point we produce every side of a convex polygon, we get an exterior angle at every vertex. This angle is the supplement of the adjacent angle in the polygon; therefore:

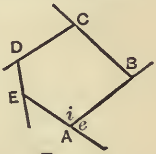


FIG. 113.

Theorem. In any convex plane polygon the sum of the exterior angles, one at each vertex, is a perigon.

243 (b). A *trapezoid* is a quadrilateral with two sides parallel.

CHAPTER XI.

PERIPHERY ANGLES.

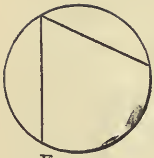


FIG. 114.

244. An *inscribed angle* is one whose arms are chords from the same point on the circle.

245. A *tanchord angle* is one between a tangent to a circle and a chord from the point of contact.

FIG. 115.

246. Inscribed angles and tanchord angles are called *periphery angles.*

247. A periphery angle is said to intercept or stand upon the arc swept over by the describing ray.

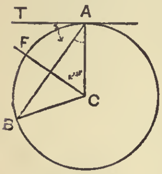


FIG. 116.

248. Theorem. A periphery angle is half the angle at the center, standing on the same arc.

Proof. Draw the bisector of the minor [reflex] angle at the center on the minor [major] arc intercepted by an acute [obtuse] tanchord angle. This is $\perp$ to the chord; $\therefore$ it makes with the radius to the point of contact an angle whose arms are $\perp$ to those of the tanchord angle; $\therefore$ both being acute [obtuse] they are equal.

An inscribed angle is the difference of two tanchord angles, and its intercepted arc is the difference of theirs; so its angle at the center is the difference of theirs.

249. Corollary I. All periphery angles on the same arc are equal.

For each is equal to half the angle at the center on this arc.

250. Corollary II. Periphery angles on explemental arcs are supplemental.

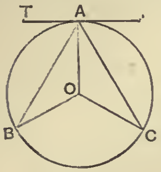


Fig. 117.

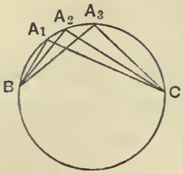


Fig. 118.

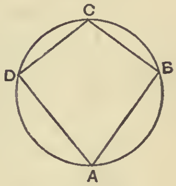


Fig. 119.

For they are halves of the explemental angles at the center.

251. Points on the same circle are called *concyclic.*

252. A polygon whose vertices are concyclic is called *cyclic.*

253. The opposite angles are supplemental in every cyclic quadrilateral (**250**).

254. In a cyclic quadrilateral any angle equals the opposite exterior angle.

255. In the same or equal circles all equal periphery angles intercept equal arcs; and inversely.

For the corresponding angles at the center are equal.

257. Theorem. An angle made by two chords is half the sum of the angles at the center standing on the arcs intercepted by it and its vertical.

Proof. $\measuredangle x = \measuredangle y + \measuredangle z.$

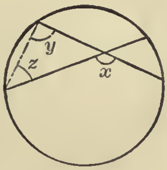


Fig. 120.

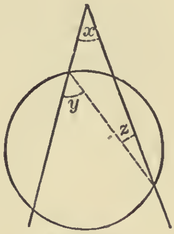


Fig. 121.

258. Theorem. An angle made by two secants is half the difference of angles at the center standing on the intercepted arcs.

Proof. $\measuredangle x = \measuredangle y - \measuredangle z.$

259. Theorem. All angles equal to a given angle, and whose arms pass through two given points, have their vertices on two symmetrical arcs ending in these points.

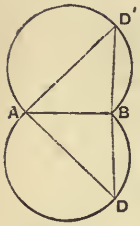


Fig. 122.

Proof. Find the center of a circle through the vertex of one such angle and the two given points, and draw the arc ending in those points and containing the vertex, and the arc ·|· to this with regard to the straight through the given points.

All angles with arms through the two given points are < the given angle if their vertices fall without this figure [258]; > if within it [257].

260. Corollary I. The vertices of all right-angled triangles on the same hypothenuse are concyclic.

261. Cor. II. If two opposite angles of a quadrilateral are supplemental, it is cyclic.

CHAPTER XII.

THE SYMMETRICAL TRIANGLE.

262. The figure consisting of three points can only be symcentral if they are in the same straight: consequently no triangle has a symcenter.

263. In any triangle a sect joining a vertex to the mid point of the opposite side is called a *median*.

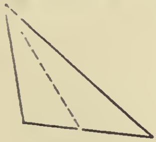


FIG. 123.

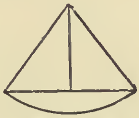


FIG. 124.

264. A perpendicular from a vertex to the opposite side is called an *altitude*.

265. The figure consisting of three points can only be symmetrical with regard to an axis passing through one and bisecting at right angles the sect joining the other two; consequently, every symmetrical triangle is isosceles, and has a median which is an altitude- and an angle-bisector.

266. If with the intersection of the equal sides of any isosceles triangle as center, and one of the sides as radius, we describe a circle, it will pass through the other two vertices.

FIG. 125.

Therefore in every isosceles triangle the median concurrent with the equal sides is an altitude- and an angle-bisector. So every isosceles triangle is symmetrical.

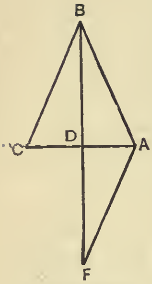


FIG. 126.

267. Theorem. A triangle having a median which is an angle-bisector is isosceles.

Proof. Produce this median BD to F, making $DF = BD$. Join AF. $\triangle ADF$ is symcentral to $\triangle DBC$; $\therefore \angle F = \angle CBD$, and $FA = BC$. But $\angle CBD = \angle DBA$;

$$\therefore \angle F = \angle DBA; \quad \therefore FA = AB; \quad \therefore AB = BC.$$

268(a). Theorem. A triangle is symmetrical if two angle-bisectors are equal.

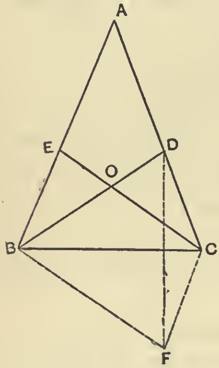


FIG. 126 (b).

Proof. If $\angle OBC$ is not $= \angle OCB$, suppose

$$\angle OBC > \angle OCB;$$

$$\therefore CD > BE. \quad (304.)$$

Now make $\angle CBF = \angle ECB$;

also $\angle BCF = \angle EBC$;

$$\therefore BF = CE,$$

and $CF = BE$.

Join DF.

Then, since $BF = BD$,

$$\therefore \angle BFD = \angle BDF;$$

and by hypoth., $\angle OCD < \angle OBE$,

and $\angle COD = \angle BOE$;

$$\therefore \angle ODC > \angle OEB,$$

and $\therefore \angle ODC > \angle BFC$.

Hence, subtracting $\measuredangle BDF = \measuredangle BFD$,

$$\therefore \measuredangle FDC > \measuredangle DFC;$$

$$\therefore CF > CD,$$

$$\therefore BE > CD,$$

$$\therefore BE > \text{ and } < CD;$$

which is absurd.

$$\therefore \measuredangle OBC = \measuredangle OCB.$$

268 (*b*). If any triangle has one of the following properties, it has all:

[1] Symmetry.
[2] Two equal sides.
[3] Two equal angles.
[4] A median which is an altitude.
[5] A median which is an angle-bisector.
[6] An altitude which is an angle-bisector.
[7] A perpendicular side-bisector which contains a vertex.
[8] Two equal angle-bisectors.

CHAPTER XIII.

THE SYMCENTRAL QUADRILATERAL.

269. A quadrilateral with a symcenter is called a *parallelogram.* (‖g'm).

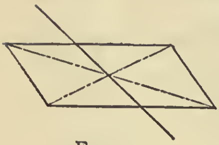


FIG. 127.

270. Because it has symcentry, every parallelogram has its opposite sides parallel and equal, its opposite angles equal, and diagonals which bisect each other. Also, every straight through the symcenter cuts the parallelogram into congruent parts.

FIG. 128.

271. Theorem. A quadrilateral with each side parallel to its opposite is a parallelogram.

Proof. Since for any two ‖s the mid point of the sect they intercept on any transversal is a symcenter, ∴ the mid point of a diagonal, being a symcenter for both pairs, is a symcenter for the quad.

272. Theorem. A quadrilateral with a pair of sides equal and parallel is a parallelogram.

Proof. The mid point of a diagonal is a symcenter for the four vertices.

273. Theorem. A quadrilateral with each side equal to its opposite is a parallelogram.

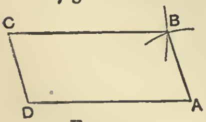


FIG. 129.

Proof. Any vertex, B, is the only intersection point of $\odot A\ [AB]$ with $\odot C\ [CB]$ on that side of the center-straight AC. But a straight through A ‖ to DC meets a straight through C ‖ to DA at that point, since opposite sides of a ‖g'm are equal.

274. Theorem. A quadrilateral with a pair of opposite sides equal and each greater than a diagonal making equal alternate angles with the other sides, is a parallelogram.

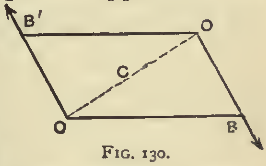


FIG. 130.

Proof. For the mid point C of this diagonal is the symcenter of its end points; and also of the other two vertices, since one of these, B, is the one intersection point of a semicircle, whose center O is one end point of this diagonal, with a ray starting in this diameter; and the other, B', is the one intersection point of a semicircle and ray symcentral to those with regard to this diagonal's mid point.

275. If the sides given equal were less than the diagonal making equal angles with the other sides, then the first ray would start from without the first semicircle and meet it twice [looking at a tangent as a secant through two coincident points].

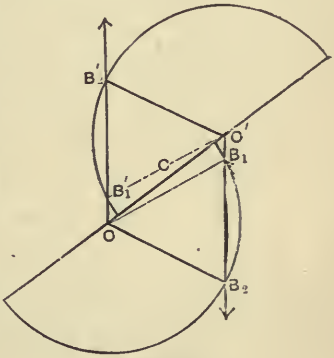


FIG. 131.

276. Theorem. A quadrilateral with a side equal to its opposite and less than a diagonal opposite equal angles is a parallelogram.

Proof. For the mid point of the diagonal is the symcenter

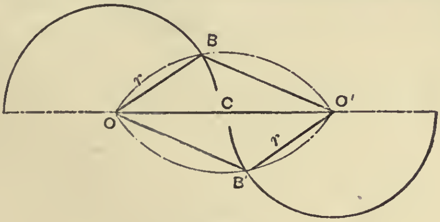


FIG. 132.

of its end points; and also of the other two vertices, since one of them, B, is the one intersection point of an arc on this

diagonal as chord with a semicircle whose center is one end point of this diagonal and radius a side less than it; and the other vertex, B', is the one intersection point of an arc and semicircle symcentral to those with regard to this diagonal's mid point.

277. A circle $\odot O$ (r), on a ray from whose center a chord OO' is, can meet that chord only once; but if it cuts the arc of that chord twice before meeting the chord, it never meets the chord.

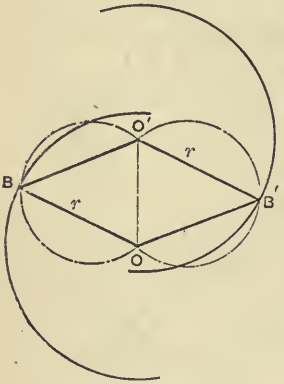


FIG. 133.

278. If the given equal sides, r, were greater than the diagonal OO' opposite the equal angles, then the first semicircle would not meet the chord of the first arc, and so would intersect that arc twice.

279. Theorem. A quadrilateral with each angle equal to its opposite is a parallelogram.

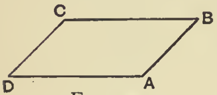


FIG. 134.

Proof. For then any two of the angles not opposite equal the other two, and therefore are supplemental. So each side is || to its opposite.

280. Theorem. A quadrilateral whose diagonals bisect each other is a parallelogram.

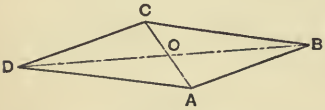


FIG. 135.

Proof. Their intersection is then a symcenter for the four vertices.

CHAPTER XIV.

SYMMETRICAL QUADRILATERALS.

281. A symmetrical quadrilateral with a diagonal as axis is called a *deltoid.*

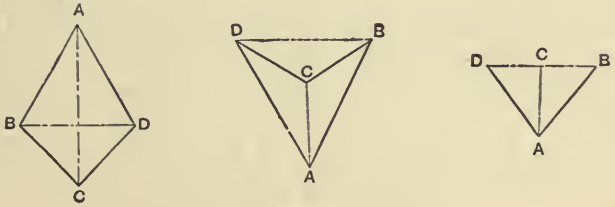


FIG. 136.

282. A sect joining the mid points of the opposite sides of a quadrilateral is called a *median.*

283. A symmetrical quadrilateral with a median as axis is called a *symtra.*

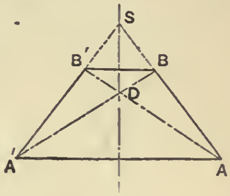


FIG. 137.

284. Theorem. Every symmetrical quadrilateral not a deltoid is a symtra.

Proof. For to every vertex corresponds a vertex, hence the number of vertices not on the axis must be even,—here four; and the sects joining corresponding vertices are bisected at right angles by the axis, hence parallel, hence sides; for the two diagonals of an undivided tetragon can never be parallel, since not every pair of conjugate angles made

by the diagonals with the four sides can be as great as a straight angle.

285. In any deltoid, since a diagonal is axis of symmetry, therefore :

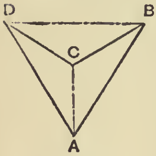


FIG. 138.

[1] One diagonal [the axis] is the perpendicular bisector of the other.

[2] One diagonal [the axis] bisects the angles at the two vertices.

[3] Sides which meet on one diagonal [the axis] are equal; so each side is equal to one of its adjacent sides.

[4] One diagonal [not the axis] joins the vertices of equal angles, and makes equal angles with the equal sides.

[5] The triangles made by one diagonal [the axis] are congruent, and their equal sides meet.

[6] One diagonal [not the axis] makes two isosceles triangles.

CONDITIONS SUFFICIENT TO MAKE A QUADRILATERAL A DELTOID.

286. Any quadrilateral which has one of the six preceding pairs of properties is a deltoid; for from [1] that diagonal is an axis of symmetry; from [2] that diagonal is axis; from [3] if $AB = AD$ and $CB = CD$, then the isosceles triangles ABD, CBD have a common axis of symmetry, AC. This follows also from [6]; from [4] the perpendicular bisector of that diagonal must be axis of symmetry for the two equal angles, and their corresponding sides must intersect on it, hence it is a diagonal; from [5] taking two adjacent sides equal, and the angle contained by them bisected by a diagonal, then the ends of these equal sides are corresponding points with regard to this diagonal as axis of symmetry.

287. Theorem. A quadrilateral with a diagonal which bisects the angle made by two sides, and is less than each of the

other two sides, and these sides equal, is a deltoid with this diagonal as axis.

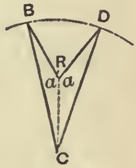


FIG. 140.

Proof. One of the two vertices not on this diagonal is the one intersection point of a semicircle whose center is one end point of that diagonal, with a ray starting in its diameter; and the other is the one intersection point of a semicircle, and ray symmetrical to those with regard to this diagonal.

288. Theorem. A quadrilateral with a side meeting an equal side in a greater diagonal which is opposite equal angles is a deltoid with that diagonal as axis.

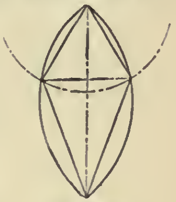


FIG. 141.

Proof. Of the two vertices not on this diagonal one is the one intersection point of an arc on this diagonal as chord with a semicircle whose center is one end point of this diagonal and radius a side less than it; and the other vertex is the one intersection point of an arc and semicircle symmetrical to those with regard to this diagonal.

289. In any symtra, since a median is axis of symmetry, therefore:

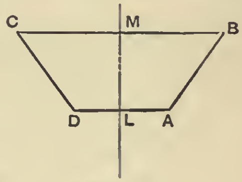


FIG. 142.

[1] Two opposite sides are parallel, and have a common perpendicular bisector.

[2] The other two sides are equal, and make equal angles with the parallel sides.

[3] Each angle is equal to one and supplemental to the other, of the two not opposite to it.

[4] The diagonals are equal, and their segments adjacent to the same parallel are equal.

[5] One median bisects the angle between the two diagonals, and also the angle between the non-parallel sides, when produced.

CONDITIONS SUFFICIENT TO MAKE A QUADRILATERAL A SYMTRA.

290. Any quadrilateral which has one of the preceding five pairs of properties is a symtra.

[1] Here the common perpendicular bisector is an axis of symmetry.

[2] Here the perpendicular bisector of the parallel sides is a symmetry axis for the four vertices.

[3] Since this is the same as two sides $\|$ and the $\measuredangle^s$ adjacent to either equal, therefore here the rt' bi' of the side joining the vertices of the equal angles is symmetry-axis for those vertices and angles, and for the parallel containing the opposite side; $\therefore$ for the intersection points of this parallel with the sides of the equal angles [the other two vertices].

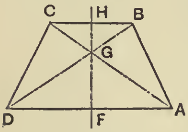


FIG. 143.

[4] Here since $GB = GC$, and $GA = GD$, therefore the bi' of $\measuredangle\ AGD$ is symmetry-axis for the four vertices.

[5] Here since in each of the triangles AGD, CGB a median bisects an angle, therefore it is symmetry axis for the vertices.

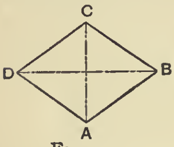


FIG. 144.

291. A symcentral deltoid is called a *rhombus.*

292. In a rhombus:

[1] All four sides are equal.

[2] Each diagonal is a symmetry axis.

[3] Each diagonal is perpendicular to the other, and bisects two angles.

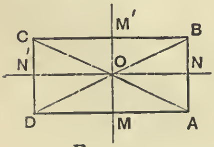


FIG. 145.

293. Inversely, a quadrilateral with [1], [2], or [3] is a rhombus.

294. A symcentral symtra is called a *rectangle.*

295. In a rectangle:

[*a*] All its angles are right.

[*b*] Each median is a symmetry axis.

[*c*] Its diagonals are equal, and bisect each other.

296. Inversely, a quadrilateral with [*a*], [*b*], or [*c*] is a rectangle.

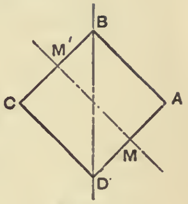


FIG. 146.

297. A symtral deltoid is called a *square*.

298. A square has symcentry, and so has a rhombus and a rectangle.

299. A quadrilateral with [1] and [*a*], or [2] and [*b*], or [3] and [*c*], is a square.

CHAPTER XV.

CONGRUENCE OF TRIANGLES.

300. Theorem. Triangles are congruent if they have a side and two angles adjacent to it equal; or a side and two angles,

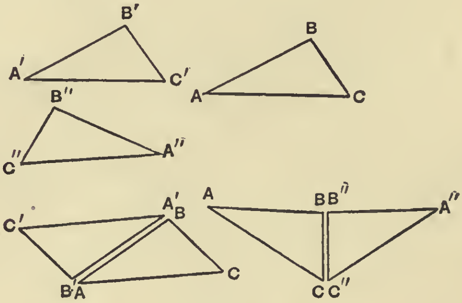


Fig. 147.

one adjacent and one opposite to it, respectively equal; or two sides and the included angle equal, or two sides and the angle opposite the greater equal, or three sides equal.

Proof. Since in any triangle the sum of the three angles is a straight angle, the second case comes under the first. In every case, slide the two triangles in the plane until a pair of equal sides coincide, but beyond this common side are no coincident points. If then a pair of equal angles have a common vertex, or a second pair of equal sides have a common end point, the triangles are symmetrical with regard to the common side. If not, symcentral with regard to its mid point.

301. Theorem. If two triangles have two sides and the angle opposite the lesser equal, they either are congruent or have supplemental angles opposite the greater equal sides.

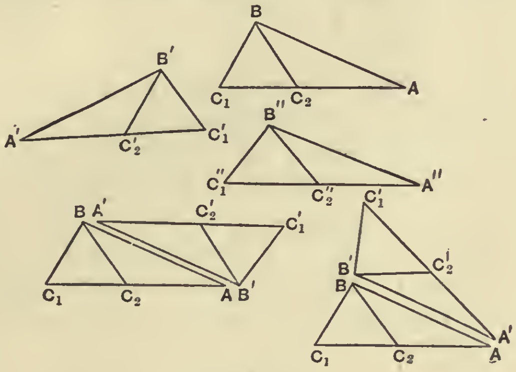


Fig. 148.

Proof. Slide the triangles in the plane until the greater equal sides coincide, but beyond this common side are no coincident points. Then in one triangle the vertex opposite the common side is one of the two intersection points of a secant from one end point with a semicircle whose center is the other end point of the common side, and in the other triangle is one of two points, which, if the angles given equal have now a common vertex, are symmetrical to these with regard to the common side; if not, symcentral with regard to its mid point. If corresponding points of these four be vertices, the triangles are congruent. If not, then opposite the common side the angle in one triangle equals the exterior angle in the other.

302. Corollary. If two triangles have two sides of the one equal respectively to the sides of the other, and the angles opposite to one pair of equal sides equal, then, if the angles opposite the other pair of equal sides are not supplemental, or if any one angle in either triangle is a right angle, the triangles are congruent.

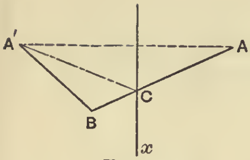


FIG. 149.

303. Theorem. Of sects joining two symmetrical points to a third, that cutting the axis is the greater.

Proof. $BA = BC + CA'$;

$$BC + CA' > BA'.$$

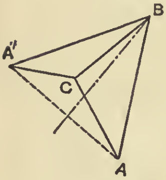


FIG. 150.

304. Theorem. If two triangles have two sides of the one respectively equal to two sides of the other, but the included angles unequal, then that third side is the greater which is opposite the greater angle.

Proof. Slide the triangles in the plane until a pair of equal sides coincide, and the other pair of equal sides have a common end point. Bisect the angle made by these equal sides. This axis cuts the third side which is opposite the greater angle.

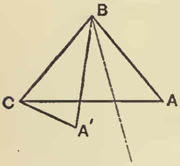


FIG. 150*a*.

305. Inverse. If two triangles have two sides of the one respectively equal to two sides of the other, but the third sides unequal, then, of the angles opposite these third sides, that is the greater which is opposite the greater third side.

306. Theorem. If three parallels intercept equal sects on one transversal they intercept equal sects on every transversal.

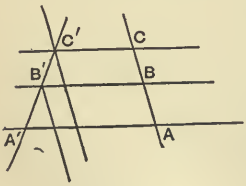


FIG. 151.

Proof. If, on a straight, $AB = BC$, and $\|^s$ through A, B, and C intersect another st' in A', B', C', then st's through B' and C', drawn $\|$ to BC, make $\cong \triangle^s$.

307. Corollary I. The intercept made on the mid of three parallels by two transversals differs from the intercepts on the others by equal sects.

308. Cor. II. If through the mid point of one side of a triangle a straight be drawn parallel to a second side, it will bisect the third side.

309. Cor. III. If a straight parallel to one side of a triangle cuts off any fractional part of a side, it cuts off the same fraction of the other side.

310. Inverse. The sect joining the mid points of any two sides of a triangle is parallel to the third side, and equal to half of it.

311. Corollary. The sect joining points which bound with any vertex of a triangle the same fractional parts of two sides is parallel to the third side and is that fractional part of it.

ROTATION-CENTER.

312. Theorem. In a plane, the result of sliding any polygon is the same as of a rotation about a fixed point.

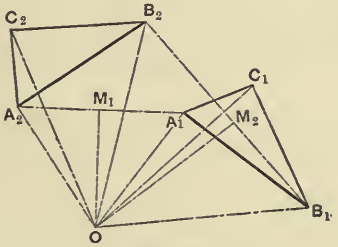


FIG. 152.

Proof. Join vertex A' with its trace A'', and B' with B''. The perpendicular bisectors of $A'A''$ and $B'B''$ intersect in the rotation-center O. For $\triangle\ A'OB' \cong \triangle\ A''OB''$ [having three sides respectively equal].

Consequently $\measuredangle\ A'OA'' = \measuredangle\ B'OB''$.

313. The altitudes of a triangle are concurrent, and the point is called the triangle's *orthocenter*.

They must cointersect, since each contains the circumcenter of a triangle made by drawing through the vertices of the given triangle parallels to its sides.

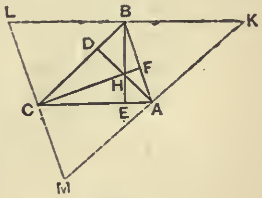


FIG. 153.

EXERCISES ON BOOK I.

1. If, with the vertex of an angle as center, two circles be described, and the points in which they cut its arms be joined, the joins are ∥ or intersect on the angle's bisector.

2. In a ∔ △ two sides, two altitudes, two medians, two ∡-bisectors are =, and cross on the axis.

3. Intersecting equal circles are ∔ with regard to their common chord.

4. If about two given points as centers pairs of equal intersecting ⊙s be described, all the pairs have their common points on one straight.

5. If of two convex polygons one is wholly within the other, then the outer has the greater perimeter.

6. An interior angle of a regular dodecagon is what fraction of a r't ∡?

7. If two sides of a △ be produced through their common vertex until each is doubled, the join of the ends is ∥ to the third side.

8. The join of the points of contact of ∥ tangents to a ⊙ is a diameter.

9. If in a ∔ △ we decrease one of the equal sides and increase the other equally, the join of the points so obtained is bisected by the third side.

10. In △ ABC, if r't bi' of a cuts the st' b in D and c in E, then is

$$\measuredangle ABD = \measuredangle ACE = \text{dif' between } \measuredangle\text{s } B \text{ and } C.$$

11. If from two p'ts of a st', ⊥s to another st' are =, either the st's are ∥, or the sects from their cross to the p'ts are =.

12. The sum of the ⊥s to the = sides from any p't in the third side of a ∔ △ equals one of the = altitudes.

13. All equal sects between two ∥s belong to two sets of ∥s.

14. If from the vertices in the same sense on the sides of a ∥g'm a given sect be taken, the points so obtained are vertices of a ∥g'm cosymcentral with the first.

15. Find the bisector of an ∡ without using its vertex.

16. A quad' with two sides ∥ and the others = is either a ∥g'm or a symtra.

17. If two sides of a quad' have a common r't bi', it is a symtra.

18. The r't bi's of the non-∥ sides of a symtra cross on the r't bi' of the other sides.

19. If the diagonals of a quad' are =, its medians are ⊥.

20. If in a trapezoid three sides are =, then the angles adjacent to the fourth side are bisected by the diagonals.

21. The sects to the intersection points of a secant from the ⊥ projections of ends of a diameter on it are =.

22. A quad' is fixed by 5 given magnitudes.

23. An n-gon is fixed by $2n - 3$ given magnitudes.

24. The bisector of an ∡ of a △ and the r't bi' of the opposite side cross on the circum-⊙.

25. The cross of an altitude (produced through its foot) with the circum-⊙ is ∤ to the orthocenter with respect to that side of the △.

26. Whether their vertex be on or within the ⊙, a pair of vertical angles together intercept the same part of a ⊙.

27. Vertical r't ∡s with vertex on or within a ⊙ intercept half of it.

28. Joining one common p't of two = intersecting ⊙s to the crosses of a secant through the other common point gives = sects.

29. If from one intersection p't of two = ⊙s as center we describe any third circle cutting them, then the four intersection p'ts are vertices of a symtra whose non-∥ sides go through the other intersection p't of the = ⊙s.

30. A ⊙ on the common chord of two = ⊙s as diameter bisects all sects through an intersection p't of the ⊙s and ending in them.

31. A symtra is cyclic.

32. A deltoid is a circumscribed quad'.

33. The four ∡-bisectors of a quad' make a cyclic quad'.

34. The four crosses of the inner with the outer common tangents to two ⊙s lie on a circle with their center-sect as diameter.

35. The sect of an outer between the inner tangents equals the sect of an inner between its points of contact.

36. Each side of a △ is, by the p'ts of contact of the in-⊙ and an ex-⊙, divided into three sects, of which the outer two are =.

37. If a polygon has a circum-⊙ and a concentric in-⊙, it is regular.

38. To make a regular hexagon, trisect the sides of a regular trigon and join the points next its vertices.

39. To make a regular octagon, about each vertex of a square, with half the diagonal as radius, describe a ⊙ and join the crosses next its vertices.

40. If a p't of its circum-⊙ be joined to the vertices of a regular △, the greatest sect equals the sum of the other two.

BOOK II.

PURE SPHERICS.

CHAPTER I.

PRIMARY CONCEPTS.

314. A circle is a closed line that will slide in its trace. Though in itself unbounded and everywhere alike, yet it is finite. On it two points starting from coincidence and moving in opposite senses will meet.

315. Every point on a circle has one other on it such that the two bisect the circle. Two such are called *opposite* points.

316. If a pair of opposite points can be kept fixed while a circle moves, it describes a surface called a *sphere.*

317. A sphere is a closed surface which will slide in its trace. Though in itself unbounded and everywhere alike, yet it is finite, being generated completely by one finite motion of a finite line.

318. ASSUMPTION I. Any figure drawn on the sphere may be moved about in the sphere without any other change.

319. *Assumed Construction* I. Through any two points, in a sphere, can be passed a line congruent with the generating line of the sphere.

In Book II. *g-line* will always mean such a line, and sect will mean a piece of it less than half.

320. ASSUMPTION II. Two sects cannot meet twice on the sphere.

If two sects have two points in common, their g-lines coincide throughout. Through two points, not opposite points of a g-line, only one distinct g-line can pass.

321. A piece of the sphere with part of a g-line as one of its boundaries, would fit all along the g-line from either side.

322. Because of the symmetry in its generation, the sphere is cut by any g-line on it into two equal parts, called *hemispheres.*

FIG. 154.

323. If one end point of a sect be kept fixed, the other end point moving in the sphere describes what is called an arc, and the sect is said to rotate in the sphere about the fixed end point. The arc is greater as the amount of rotation is greater.

FIG. 155.

324. Two sects from the same point, when looked at with special reference to the amount of rotation necessary to bring their g-lines into coincidence, are said to form a spherical angle. The spherical angle is greater as the amount of rotation is greater.

325. When a sect has rotated just sufficiently to fall again into the same g-line, the angle described is called a straight angle, and the arc described is called a semicircle.

326. Half a straight angle is called a right angle.

327. The whole angle about a point in the sphere, that is, the angle described by a sect rotating until it coincides with its trace, is called a perigon; the whole arc is called a circle.

The fixed end point is called a *pole* of the circle, and the sect is called a spherical radius of the circle.

328. *Assumed Construction* II. A circle can be described from any pole, with any sect as radius.

329. ASSUMPTION III. All straight angles are equal.

330. Corollary I. All perigons are equal.

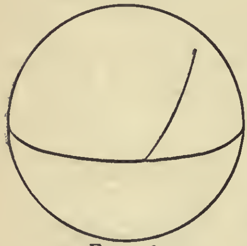


FIG. 156.

331. Corollary II. The two angles on the same side of a g-line, made by a sect with one extremity in that g-line, are together a straight angle.

332. Corollary III. Vertical angles are equal, being supplements of the same angle.

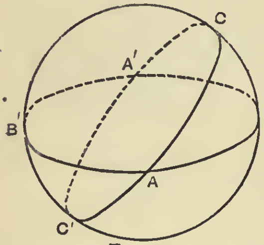


FIG. 157.

333. Theorem. Every g-line in the sphere cuts every other in two opposite points.

Proof. Let BB' and CC' be any two g-lines. Since each bisects the sphere, therefore the second cannot lie wholly in one of the hemispheres made by the first, therefore they intersect at two points, which are therefore opposite.

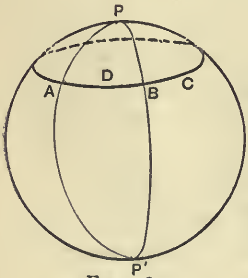


FIG. 158.

334. On a sphere, every circle has two poles, which are opposite points, and its spherical radius to one pole is the supplement of that to the other.

335. A spherical figure made by two half g-lines intersecting in opposite points, is called a *lune*.

336. Theorem. The angle contained by the sides of a lune at one of their points of intersection equals the angle contained at the other.

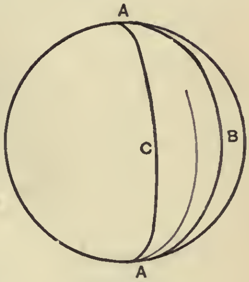


FIG. 159.

Proof. Slide the lune, in the sphere, until each of the two intersection points falls on the trace of the other, and one of the half g-lines on the trace of the other. If the angles were unequal, the smaller could thus be brought within the trace of the greater, and its second half g-line would start between the traces, and since it could meet neither again until it reached the opposite intersection point, we would find the surface of the lune less than its trace.

337. One quarter of a g-line is called a quadrant.

338. A spherical polygon is a closed figure, in the sphere, bounded by sects, no two of which cross.

339. A spherical triangle is a three-sided spherical polygon, with no interior angle greater than a straight angle.


FIG. 160.

340. A spherical triangle is positive [+] if a sect with one end pivoted within it and rotating counter-clockwise, after passing through the vertex of the greatest angle goes next over the vertex of the least.

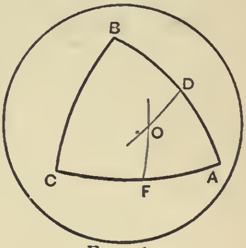


FIG. 161.

CHAPTER II.

SYMCENTRY ON THE SPHERE.

341. On a sphere a point has, with regard to a given symcenter, always one and only one symcentral point, namely, the one which ends the sect from the given point bisected by the symcenter.

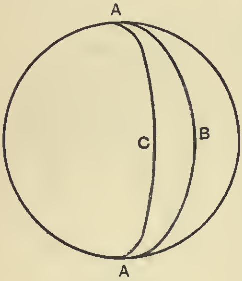


Fig. H.

342. Two figures are symcentral when they can be placed so as to have a symcenter.

One figure is symcentral when it has a symcenter, that is, a point with respect to which every point of the figure has its symcentral point on the figure.

A lune is symcentral with regard to the cross of the g-line bisecting its angles, with the g-line bisecting its sides.

343. Symcentral figures on a sphere have precisely the same properties as in the plane, including congruence.

CHAPTER III.

SYMMETRY ON THE SPHERE.

344. Two points on a sphere are symmetrical with respect to a g-line, when it bisects at right angles the sect joining them.

This g-line is called their axis of symmetry.

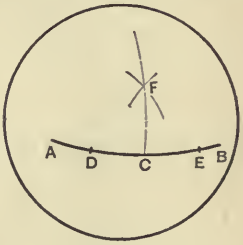


Fig. 162.

345. Two points on a sphere have always one, and only one, symmetry axis on that sphere.

346. A point has, with regard to a given axis of symmetry, always one, and only one, symmetrical point, namely, the one which ends the sect from the given point perpendicular to the axis and bisected by the axis.

347. Two figures on the sphere have an axis of symmetry when, with regard to this g-line, every point of each has its symmetrical point on the other.

Fig. 163.

348. Two figures are symmetrical when they can be placed so as to have an axis of symmetry.

349. One spherical figure has an axis of symmetry when, with regard to this g-line, every point of the figure has its symmetrical point on the figure.

350. One figure is symmetrical when it has an axis of symmetry.

351. Any figure on the sphere has, with regard to any g-line on the sphere as axis, always one, and only one, symmetrical figure.

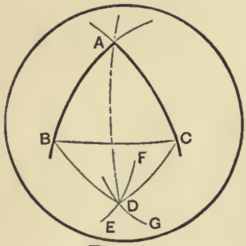


Fig. 164.

352. One figure is symmetrical when it has an axis with regard to which its symmetrical figure coincides with itself.

353. Every point on an axis is symmetrical to itself.

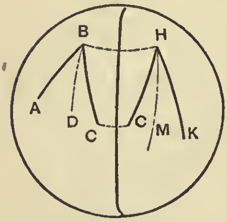


Fig. 165.

354. Assumption IV. The figure symmetrical to a sect is an equal sect; to a spherical angle, is a spherical angle equal in magnitude but opposite in sense.

355. Corollary. A sect, or g-line, or spherical angle, in one of two symmetrical figures, has a symmetrical sect, or g-line, or spherical angle, in the other.

356. The intersection point of two sects is symmetrical to the intersection of two symmetrical to those.

357. The intersection points of two symmetrical g-lines are on the axis.

358. The bisector of a spherical angle is symmetrical to the bisector of the symmetrical spherical angle.

359. The angle between two symmetrical g-lines is bisected by the axis.

360. Two g-lines are symmetrical with regard to either of their angle-bisectors.

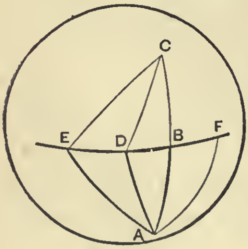


Fig. 166.

For there is a g-line symmetrical to the first with regard to that angle-bisector, and the angle between the two symmetrical g-lines is bisected by the axis.

361. Any g-line is symmetrical with regard to any of its perpendiculars.

362. Any circle is symmetrical with regard to any of its spherical diameters.

363. Every point on the perpendicular bisector of a sect is the pole of a circle passing through its end points.

For $A \cdot|\cdot B$; axis CD; $\therefore CA = CB$.

Thus sects from any point on its perpendicular bisector to the end points of the sect are equal.

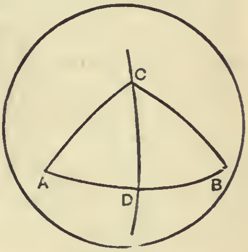


FIG. 167.

364. The perpendicular bisector of a spherical chord contains the poles of the circle. For the end points of the chord are symmetrical with regard to this perpendicular, and also with regard to the perpendicular from a pole.

365. Two circles with three points in common coincide.

366. One spherical radius, of every circle on the sphere, is less than a quadrant.

Call its pole the q-pole, and it the q-radius.

367. If the q-pole-sect of two circles equals the sum of their q-radii, they have a common point on their q-pole-sect, and by symmetry no other common point.

Such circles are said to be tangent externally.

Neither has a point in common with a circle concentric with the other, but of lesser q-radius.

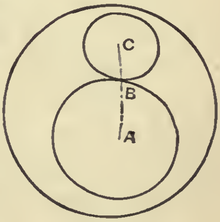


FIG. 168.

368. If the q-pole-sect equals the difference of the q-radii, the two circles have a common point on their pole-g-line, and by symmetry, no other common point.

Such circles are said to be tangent internally.

Neither has a point in common with a circle concentric with the lesser and of lesser q-radius.

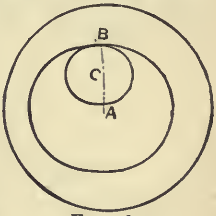


FIG. 169.

369. While the q-pole-sect is growing, from equality with the difference of the q-radii, up to equality with their sum, the two circles have always two common points, symmetrical with regard to their pole-g-line.

370. Problem. To make a spherical triangle, given its sides.

Construction. If two of its sides are each less than a quadrant, then with these as q-radii, and the end points of the third side as poles, describe two circles. Their two common points will be the third vertices of two symmetrical triangles with the three given sides.

If two of the given sides are each greater than a quadrant, take, in the above, their supplements with the given third side. Then in the two triangles obtained, produce these two supplements until they meet.

These two meeting points will be the third vertices of two symmetrical triangles with the three given sides.

371. Corollary I. Any two sides of a spherical triangle are together greater than the third.

For if two be each less than a quadrant, and together equal to the third, the construction circles will be tangent extenally.

If two be each greater than a quadrant, their difference is that of their supplements, which is less than the third side; for if equal to it, the construction circles would be tangent internally.

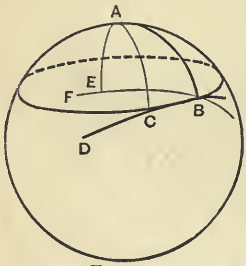


Fig. 170.

372. Corollary II. The sum of the three sides of any spherical triangle is less than a g-line.

373. Since any chord is bisected by the perpendicular from a pole, ∴ a g-line $\perp$ to a diameter at an end point has only this point in common with the circle.

This point of the circle is symmetrical to itself with regard to this diameter as axis.

But if we draw through this point B any g-line BF not $\perp$ to

the spherical radius AB, then the perpendicular from a pole A will meet this g-line BF at some other point E.

Hence the g-line BF cuts the circle again at $B' \cdot|\cdot B$, axis AE; $\therefore$

Theorem. At every point on the circle one, and only one, tangent can be drawn, namely, the perpendicular to a radius at that point.

374. Let P be a point not in the g-line g, and $PC \perp$ to g: then g is tangent to $\odot P[PC]$ at C.

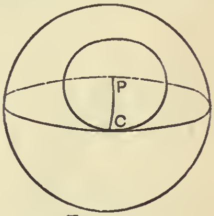


FIG. 171.

If PC is less than a quadrant, any second circle with q-radius $< PC$, and q-pole P, lies wholly within $\odot P[PC]$. Therefore:

Theorem. If less than a quadrant, the perpendicular is the least sect between a point and a g-line.

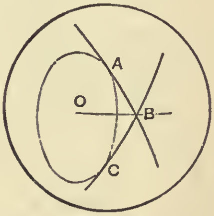


FIG. 172.

375. The poles of all circles tangent to two intersecting g-lines are in their angle-bisectors.

376. From any point on an angle-bisector the perpendiculars to the g-lines are equal.

377. Theorem. If two angles of a triangle be equal, the triangles is isosceles.

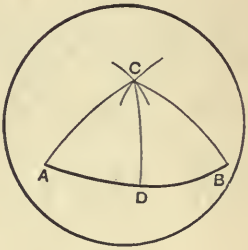


FIG. 173.

Proof. The perpendicular bisector of the side joining the equal angles is the symmetry axis for that side and its end points, and so for angles made with that side at those points which are equal in magnitude and opposite in sense.

378. Theorem. If one angle of a spherical triangle is greater than a second, the side opposite the first must be greater than the side opposite the second.

Proof. Given the $\measuredangle C > \measuredangle A$.

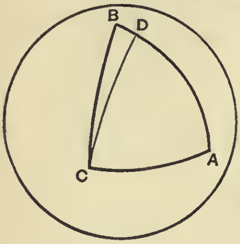


FIG. 174.

If $\measuredangle DCA = \measuredangle A$, then $DC = DA$.

But $DC + DB > BC$; $\therefore DA + DB > BC$.

379. Inverse. If one side of a spherical triangle is greater than a second, the angle opposite the first must be greater than the angle opposite the second.

Proof. For the angle opposite the second cannot be the greater, nor can they be equal.

380. Theorem. In an isosceles triangle the angles opposite the equal sides are equal.

Proof. The bisector of the angle between the equal sides is a symmetry axis for those sides and their end points, hence for the triangle.

381. Corollary. In an isosceles triangle the bisector of the angle between the equal sides is perpendicular to the third side.

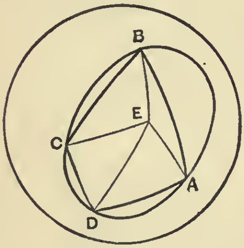


FIG. 175.

382. If the vertices of a polygon are concyclic, the polygon may be called *cyclic.*

383. In a cyclic quadrilateral, the sum of one pair of opposite angles equals the sum of the other pair.

Proof. Join the circumcenter E with A, B, C, D, the vertices. By isosceles triangles, $\measuredangle ABC = \measuredangle BAE + \measuredangle ECB$, and $CDA = \measuredangle DCE + \measuredangle DAE$.

CHAPTER IV.

THE SYMCENTRAL QUADRILATERAL.

384. A symcentral spherical quadrilateral, or *cenquad*, has its opposite sides equal, its opposite angles equal, and diagonals which bisect each other.

Also, every g-line through the symcenter cuts the cenquad into congruent parts.

385. Theorem. A quadrilateral with a diagonal making with each side an angle equal to its alternate, is a cenquad.

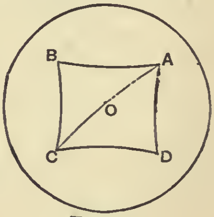


Fig. 176.

Proof. The mid point of this diagonal is a symcenter for both pairs of opposite sides.

386. Theorem. A quadrilateral with a pair of opposite sides equal and making equal alternate angles with a diagonal, is a cenquad.

Proof. The mid point of the diagonal is a symcenter for the four vertices.

387. Theorem. A quadrilateral with a pair of opposite sides equal, and a diagonal making equal alternate angles with the other sides and opposite angles not supplemental, is a cenquad.

Proof. The mid point of this diagonal is the symcenter of its end points; and also of the other two vertices, since one of these is an intersection point of a semicircle, of which a diameter is bisected by one end point of this diagonal, with a g-line through its other end; and the other is the symcentral inter-

section point of a semicircle and g-line symcentral to those with regard to this diagonal's mid point.

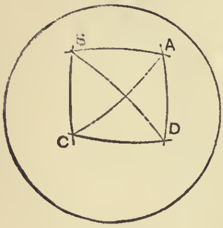


FIG. 177.

388. Theorem. A quadrilateral with each side equal to its opposite is a cenquad.

Proof. Any vertex, B, is the only intersection point of $\odot A$ $[AB]$ with $\odot C$ $[CB]$ on that side of their pole-g-line, AC.

But the fourth vertex of a cenquad with sides $CD = AB$ and $DA = CB$, and symcenter the mid point of AC, is that point B.

389. Theorem. A quadrilateral whose diagonals bisect each other is a cenquad.

Proof. Their intersection is then a symcenter for the four vertices.

CHAPTER V.

SPHERICAL TRIANGLES.

390. Theorem. Spherical triangles of the same sense are congruent if they have a side and two angles adjacent to it equal; or two sides and the included angle equal; or two sides and the angles opposite one pair equal, opposite the other pair not supplemental; or three sides equal.

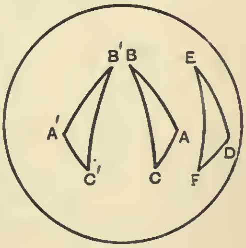


Fig. 179.

Proof. Slide the two triangles in the sphere until a pair of equal sides coincide, but beyond this common side are no coincident points. The triangles are then symcentral with regard to the mid point of the common side.

391. Triangles which would be congruent, but that they differ in sense, are symmetrical. Symmetrical triangles are of opposite sign.

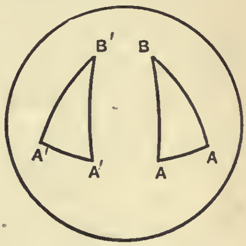


Fig. 180.

392. Corollary. Symmetrical isosceles spherical triangles are congruent.

For the equality of two angles in a triangle obliterates the distinction of sense or sign.

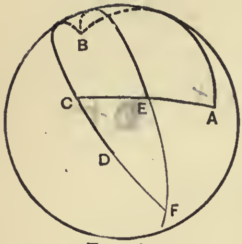


FIG. 181.

393. Theorem. An exterior angle of a spherical triangle is greater than, equal to, or less than either of the interior opposite angles, according as the median from the other interior opposite angle is less than, equal to, or greater than a quadrant.

Proof. Let ACD be an exterior angle of the $\hat{\triangle}$ ABC. Bisect AC at E. Join BE, and produce to F, making $EF = BE$. Join FC.

$$\hat{\triangle} ABE \cong \hat{\triangle} CFE.$$

[Spherical triangles of the same sense having two sides and the included angle equal are congruent.]

$$\therefore \measuredangle BAE = \measuredangle FCE.$$

If, now, the median BE be a quadrant, BEF is a half-g-line, and F lies on BD; $\therefore$ $\measuredangle DCE$ coincides with $\measuredangle FCE$, $\therefore$ $\measuredangle DCE = \measuredangle BAE$.

If the median BE be less than a quadrant, BEF is less than a half-g-line, and F lies between CD and AC; $\therefore$ $\measuredangle DCA > \measuredangle FCE$, $\therefore$ $DCA > \measuredangle BAC$.

And if BE be greater than a quadrant, BEF is greater than a half-g-line, and F lies between CD and AC produced through C; $\therefore$ $\measuredangle DCA < \measuredangle FCE$, $\therefore$ $DCA < \measuredangle BAC$.

Thus, according as BE is greater than, equal to, or less than a quadrant, the exterior $\measuredangle ACD$ is less than, equal to, or greater than, the interior opposite $\measuredangle BAC$.

394. Inversely, according as the exterior angle ACD is greater than, equal to, or less than the interior opposite angle BAC, the median BE is less than, equal to, or greater than a quadrant.

395. Theorem. Any two perpendiculars to a g-line intersect in two points, from either of which all sects drawn to that g-line are quadrants perpendicular to it.

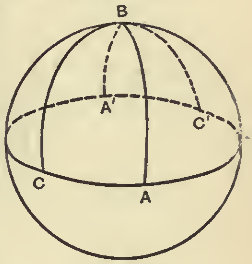


FIG. 182.

Proof. Let AB and CB, drawn at right angles to AC, intersect at B, and meet AC again at A' and C', respectively.

Then $\measuredangle BA'C' = \measuredangle BAC'$ and $\measuredangle BC'A' = \measuredangle BCA'$.

[The angles contained by the sides of a lune, at their two points of intersection, are equal.]

Moreover, $AC = A'C'$, for they have the common supplement AC'. Hence, keeping A and C on the line AC, slide ABC until AC comes into coincidence with $A'C'$. Then the angles at A, C, A', C' being all right, AB will lie along $A'B$, and CB along $C'B$, and hence the figures ABC and $A'BC'$ coincide.

Therefore each of the half-lines ABA' and CBC' is bisected at B.

In like manner, any other line drawn at right angles to AC passes through B, the mid point of ABA'.

Hence every sect from AC to B is a quadrant $\perp$ to AC.

396. Corollary I. A g-line is a circle whose spherical radius is a quadrant.

397. Corollary II. A point which is a quadrant from two points in a g-line, and not in the g-line, is its pole.

398. Corollary III. Equal angles at the poles of lines intercept equal sects on those lines.

399. The *polar* of any point is the g-line of which that point is a pole.

400. If an angle be a fraction of a perigon, it intercepts on the polar of its vertex that fraction of a g-line.

401. Theorem. If a median is a quadrant, it is an angle-bisector, and the sides of the bisected angle are supplemental.

Proof. The quadrant and the sides BA, BC, all produced, are concurrent in B' opposite to B. $\therefore ABCB'A$, is a cenquad [its diagonals AC and BB' bisect each other]. $\therefore AB = CB'$, and AB, BC are supplemental. Also, the complements of AB and CB' are equal, i.e., $AH = CF$; also, the supplements of $\measuredangle BAD$ and $\measuredangle B'CD$, or $\measuredangle DAH = \measuredangle DCF$; $\therefore \hat{\triangle} ADH \cong \hat{\triangle} CDF$ [two sides and included angle]; $\therefore HD = DF$; $\therefore \measuredangle ABC$ is bisected by BD.

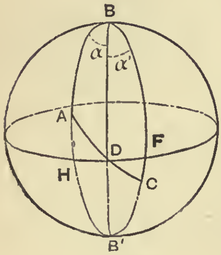


FIG. 183.

402. Corollary. If two sides of a triangle are supplemental, the opposite angles are supplemental.

403. Theorem. Two spherical triangles, of the same sense, having two angles of the one equal to two angles of the other, the sides opposite one pair of equal angles equal, and those opposite the other pair not supplemental, are congruent.

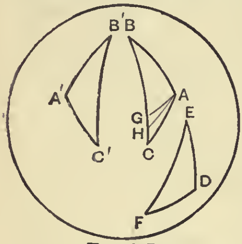


FIG 184.

Proof. Given $\measuredangle B = \measuredangle E$; $\measuredangle C = F$; $AB = DE$; AC not supplemental to FD. Make DE coincide with AB: then EF will lie along BC, and FD must coincide with AC; else would it make a $\hat{\triangle}\ AGC$ with exterior $\measuredangle\ AGB =$ interior opposite $\measuredangle\ C$, and $\therefore$ with median a quadrant, and $\therefore$ with AC supplemental to AG, that is, with AC supplemental to FD.

404. Theorem. Two spherical triangles of the same sense, having in each one, and only one, right angle, equal hypothenuses, and another side or angle equal, are congruent.

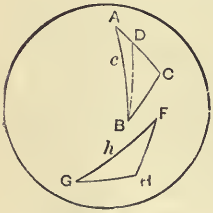


FIG. 185.

Proof. If $\measuredangle C = \measuredangle H =$ r't $\measuredangle$, and $c = h$, and $a = f$, then if $AC > g$, make $CD = g$; $\therefore BD = h = c$, and the bisector of $\measuredangle DBA$ is $\perp$ to CDA, $\therefore B$ is pole to CDA, $\therefore \measuredangle\ A$ is also r't.

If $\measuredangle C = \measuredangle H =$ r't $\measuredangle$, and $c = h$, and $\measuredangle A = \measuredangle F$, then if $\measuredangle ABC > \measuredangle G$, make $\measuredangle ABD = \measuredangle G$, $\therefore \measuredangle BDA = \measuredangle H = \measuredangle C =$ r't $\measuredangle$, $\therefore B$ is pole to CDA.

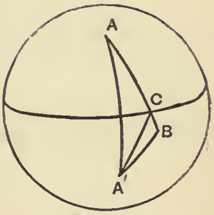


FIG. 186.

405. Theorem. Of sects joining two symmetrical points to a third, that cutting the axis is the greater.

Proof. $BA = BC + CA = BC + CA' > BA'$.

406. Theorem. If two spherical triangles have two sides of the one equal to two sides of the other, but the included angles unequal, then that third side is the greater which is opposite the greater angle.

Proof. Slide the triangles in the sphere until a pair of equal sides coincide and the other pair of equal sides have a common end point. Bisect the angle made by these equal sides. This axis cuts the third side, which is opposite the greater angle.

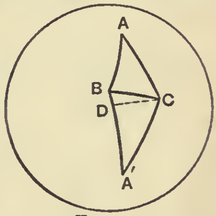


FIG. 187.

407. Inverse. If two triangles have two sides of the one equal to two sides of the other, but the third sides unequal, then of the angles opposite these third sides that is the greater which is opposite the greater third side.

408. Theorem. The g-line through the poles of two g-lines is the polar of their intersection points.

Proof. If A and B are poles of the g-lines a and b, which intersect in P, then AP and BP are quadrants; $\therefore AB$ is the polar of P.

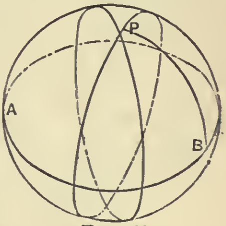


FIG. 188.

409. Corollary I. The g-line through the poles of two g-lines cuts both at right angles.

410. Corollary II. If three g-lines are concurrent, their poles are collinear.

411. Of the sides of a spherical angle, we may call those

poles positive from which in the figure these sides would be described from the vertex by a quadrant rotating positively.

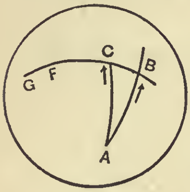


FIG. 189.

412. Theorem. The sect which an angle intercepts on the polar of its vertex equals the sect between the positive poles of its sides.

Proof. Slide the quadrant BF along the polar of A until B comes to C. The + pole F of AB will then coincide with the + pole G of AC.

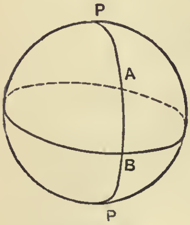


FIG. 190.

413. The sect joining any point to one pole of a g-line is less than a quadrant if the two points are in the same one of that g-line's hemispheres; greater than a quadrant if they are in different hemispheres.

By a pole's hemisphere we mean that one of its g-line's hemispheres in which the pole is.

414. Of a given spherical triangle ABC, the *polar* is a new triangle $A'B'C'$, where A' is that pole of BC which has A in its hemisphere, and B' that pole of AC which has B in its hemisphere, and C' that pole of AB which has C in its hemisphere.

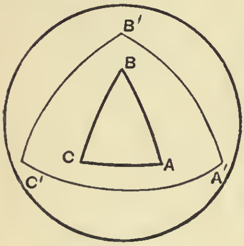


FIG. 191.

415. Theorem. If of two spherical triangles the second is the polar of the first, then the first is the polar of the second.

Hypothesis. Let ABC be the polar of $A'B'C'$.

Conclusion. Then $A'B'C'$ is the polar of ABC.

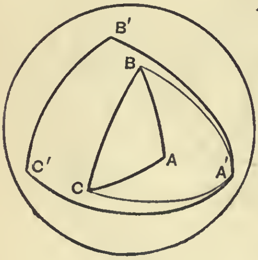


FIG. 192.

Proof. Join $A'B$ and $A'C$. Since B is pole of $A'C'$, therefore BA' is a quadrant; and since C is pole of $A'B'$, therefore CA' is a quadrant; $\therefore$ A' is pole of BC.

In like manner, B' is pole of AC, and C' of AB.

Moreover, since A has A' in its hemisphere, $\therefore$ the sect AA' is less than a quadrant, $\therefore$ A' has A in its hemisphere.

416. Theorem. In a pair of polar triangles, any angle of either intercepts, on the side of the other which lies opposite to it, a sect which is the supplement of that side.

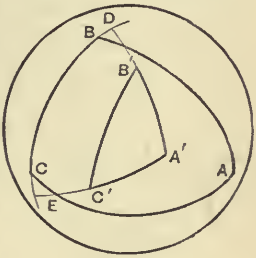


FIG. 193.

Proof. Let ABC and $A'B'C'$ be two polar triangles.

Produce $A'B'$ and $A'C'$ to meet BC at D and E, respectively. Since B is the pole of $A'C'$, therefore BE is a quadrant; and since C is the pole of $A'B'$, therefore CD is a quadrant; therefore $BE + CD =$ half-g-line; but $BE + CD = BC + DE$. Therefore DE, the sect of BC which A' intercepts, is the supplement of BC.

417. Theorem. Two spherical triangles of the same sense, having three angles of the one equal respectively to three angles of the other, are congruent.

Proof. Since the given triangles are respectively equiangular their polars are respectively equilateral.

For equal angles at the poles of g-lines intercept equal sects on those lines; and these equal sects are the supplements of corresponding sides. Hence these polars, having three sides equal, are respectively equiangular, and therefore the original triangles are respectively equilateral.

418. Of a convex spherical polygon $ABCD\ldots$, the polar is a new spherical polygon $A'B'C'D'\ldots$, where A' is that pole of BC which has A in its hemisphere, etc.

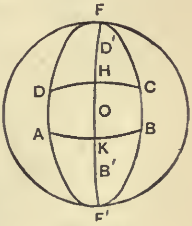


FIG. 194.

419. Theorem. The polar of a cenquad is a concentric cenquad.

Proof. The g-line HK through the symcenter O and $\perp$ to AB is also $\perp$ to CD; and $OH = OK$ are the complements of the sects from O to poles D' and B' of the sides AB and CD.

Hence O is symcenter for B' and D'.

In the same way prove O symcenter for A' and C'.

420. Theorem. The opposite sides of a cenquad intersect on the polar of its symcenter.

Proof. O is symcenter for F and F'.

421. Theorem. Any two consecutive vertices of a cenquad and the opposites of the other two are concyclic.

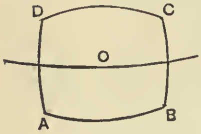


Fig. 195.

Proof. The perpendiculars, to the g-line through O and bisecting BC and DA, from A and B in one of its hemispheres and C and D in the other, are equal. So also the perpendiculars from their opposites D' and C' in the first hemisphere and A' and B' in the second.

So A, B, C', D' and A', B', C, D are on equal circles with opposite q-poles.

Such circles are called parallels; the co-polar g-line, *equator*.

422. The perpendiculars erected at the mid points of the sides of a spherical triangle are concurrent in its circumcenter.

423. Theorem. The g-line bisecting two sides of a triangle intersects the third side at a quadrant from its mid point.

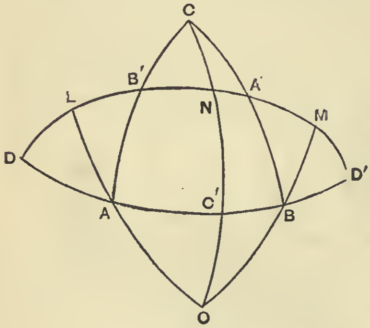


Fig. 196.

Proof. AL, BM, CN are $\perp$ to the g-line through A', B', the mid points of two sides BC, CA, and meeting the third side produced at D and D'. $\therefore ALB' \cong CNB'$ [having the right angle, hypothenuse, and one oblique angle equal], $\therefore AL = CN$.

Similarly, $BM = CN$.

$\therefore ALD \cong BMD'$ [having two angles and an opposite side equal, and the other pair of opposite sides not supplemental].

$\therefore AD = BD'$, $\therefore DC'$ a quadrant.

424. Corollary I. The altitudes of a spherical triangle are concurrent in its orthocenter.

For, regarding $A'B'C'$ as the triangle, the perpendicular to DC' at C' is the polar of D, and $\therefore \perp$ to $A'B'$.

Similarly, the perpendicular to BA' at A' is $\perp$ to $B'C'$, etc.

So the three altitudes of $A'B'C'$ are concurrent in the circumcenter of ABC.

425. Cor. II. The vertices of spherical triangles of the same angle sum on the same base are on a circle co-polar with the g-line bisecting their sides.

For $AO = BO$, $\measuredangle\, OAB = \measuredangle\, OBA$, $\measuredangle\, LAB = \measuredangle\, MBA = \frac{1}{2}[A + B + C]$.

Hence AOB is fixed, and $\therefore OC$ [supplemental to OA].

426. Theorem. The g-lines through the corresponding vertices of a triangle and its polar are concurrent in the common orthocenter of the two triangles.

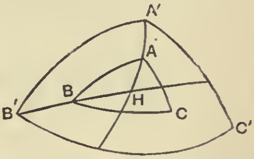


FIG. 197.

Proof. For AA' is $\perp$ to BC and $B'C'$, since it passes through their poles.

427. Theorem. The sides of a triangle intersect the corresponding sides of its polar on the polar of their orthocenter.

Proof. For AA' is the polar of the intersection points of BC and $B'C'$; similarly, BB' is the polar of the intersection points of CA and $C'A'$, etc.

Sects from the orthocenter to these intersection points are all quadrants.

428. Theorem. A triangle's in-center is also its polar's circumcenter; and R is complemental to r.

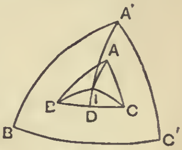


FIG. 198.

Proof. $ID \perp$ to BC contains A'. $\therefore IA'$ is the complement of r. So is IB' and IC'.

EXERCISES ON BOOK II.

1. Explemental ∡s at the q-poles of = ⊙s intercept explemental arcs.

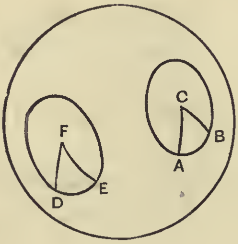


FIG A.

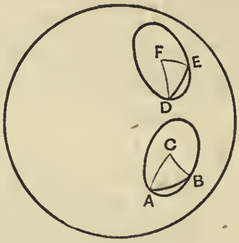


FIG. B.

2. Explemental arcs of equal circles have equal spherical chords.

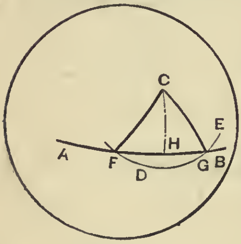


FIG. C.

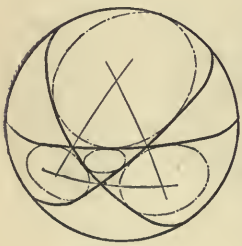


FIG. D.

3. As a spherical chord increases its major arc decreases.

4. If ⊙s pass through 2 given p'ts their centers all lie on the r't bi' of the join of the 2 p'ts.

5. If 2 ⊙s touch internally, a ⊥ to the diameter through the p't of contact has equal pieces between the 2 ⊙s.

6. The g-lines on which ⊥s from a fixed p't are equal envelop a ⊙ with this p't for center.

7. The centers of ⊙s touching two given g-lines all lie on the bisectors of the ∡s made by these g-lines.

8. The centers of ⊙s touching 3 given g-lines lie on the bisectors of the ∡s made by these g-lines.

9. If a quad' is cyclic, the r't bi's of its sides and of its two diagonals are concurrent.

10. *ABCD* is a cyclic quad'; *AD*, *BC* meet in *F*. Where does tan at *F* to circum-⊙ *CDF* meet *AB*?

11. One convex polygon wholly contained within another has the lesser perimeter.

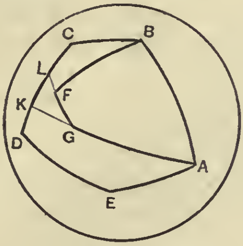


FIG. E.

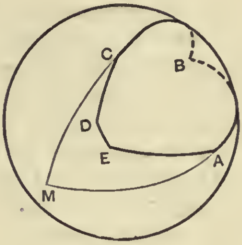


FIG. F.

12. The perimeter of any $\widehat{\triangle}$ is less than a g-line.

The perimeter of any convex spherical polygon is less than a g-line.

13. If 2 $\odot$s touch, and through the p't of contact a g-line be drawn to cut the $\odot$s again, where will the tangents at these crosses meet?

14. If 2 $\odot$s touch, and through the p't of contact 2 g-lines be drawn cutting the $\odot$s again, where will the joins of these crosses meet?

15. If the common chord of 2 intersecting $\odot$s be produced to any p't, the tangents to the 2 $\odot$s from this p't are $=$; and inversely.

16. If the common chord of 2 intersecting $\odot$s be produced to cut a common tangent, it bisects it.

17. The 3 common chords of 3 $\odot$s which intersect each other are concurrent.

18. How do the in-, circum-, and ex-radii of a regular $\widehat{\triangle}$ compare in size?

19. If a quad' can have a circle inscribed in it, the sums of the opposite sides are equal.

20. If two equal $\odot$s intersect, each contains the orthocenters of $\triangle$s inscribed in the other on the common chord as base.

21. Three equal $\odot$s intersect at a p't H, their other points of intersection being A, B, C. Show that H is orthocenter of $\triangle ABC$; and that the triangle formed by joining the centers of the circles is $\cong$ to $\triangle ABC$.

22. The feet of $\perp$s from A of $\triangle ABC$ on the external and internal bi's of ∡s B and C are co-st' with the mid p'ts of b and c.

Does this hold for the sphere?

23. If two opposite sides of a quad' are $=$, they make $=$ ∡s with the median of the other sides. Prove for the plane, then extend to the sphere.

24. (Bordage.) The centroids of the 4 $\triangle$s determined by 4 concyclic p'ts are concyclic.

25. The orthocenters of the 4 $\triangle$s determined by 4 concyclic p'ts A, B, C, D are the vertices of a quad' $\cong$ to $ABCD$. The in-centers are vertices of an equiangular quad'.

26. (Brahmegupta.) If the diagonals of a cyclic quad' are $\perp$, the $\perp$ from their cross on one side bisects the opposite side.

27. If the diagonals of a cyclic quad' are $\perp$, the feet of the $\perp$s from their cross on the sides and the mid p'ts of the sides are concyclic.

28. If tangents be drawn at the ends of any two diameters, what sort of a quad' is circumscribed?

29. In any equiangular polygon inscribed in a $\odot$, each side is equal to the next but one to it.

Hence, if an equiangular polygon inscribed in a $\odot$ have an odd number of sides it must be equilateral.

Any equilateral polygon inscribed in a $\odot$ is equiangular.

30. In any equilateral polygon circumscribed about a $\odot$, each $\measuredangle$ is $=$ to the next but one to it.

Hence, if an equilateral polygon circumscribed about a circle have an odd number of sides, it must be equiangular.

Any equiangular polygon described about a $\odot$ is equilateral.

31. The circle through any 3 vertices of a regular polygon contains the remaining vertices.

32. If one of 2 equal chords of a $\odot$ bisects the other, then each bisects the other.

33. Given 2 symcentral g-lines and their symcenter. Find the g-line symcentral to a third given g-line with respect to this symcenter.

34. All $=\widehat{\triangle}^s$ on the same side of the same base have their sides bisected by the same g-line.

BOOK III.

EQUIVALENCE.

429. Magnitudes are equivalent which can be cut into parts congruent in pairs.

430. Problem. To describe a rectangle, given two consecutive sides.

Construction. Draw a straight, erect to it a perpendicular. From the vertex of the right angle lay off one given sect on the straight, the other on the perpendicular. Through their second end points draw parallels, one to the straight, one to the perpendicular.

431. Corollary. A rectangle is completely determined by two consecutive sides; so if two sects, a and b, are given, we may speak of the rectangle of a and b, or we may call it the rectangle ab. Thus, when a and b are actual sects, we mean by ab a definite plane figure with four right angles, four sides, and an enclosed surface.

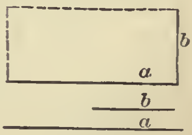


FIG. 199.

432. The sum of two polygons is any polygon equivalent to them.

433. Theorem. In any right-angled triangle, the square on the hypothenuse is equivalent to the sum of the squares on the other two sides.

Hypothesis. $\triangle$ ABC, r't angled at B.

Conclusion. Square on $AB +$ sq' on $BC =$ sq' on AC.

Proof. By 430, on hypothenuse AC, on the side toward the $\triangle$ ABC, describe the sq' $ADFC$.

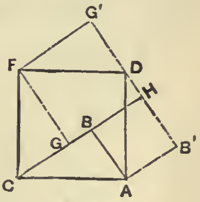


FIG. 200.

On the greater of the other two sides, as BC, lay off $CG = AB$. Join FG. Then, by construction, $CA = FC$, and $AB = CG$, and $\measuredangle CAB = \measuredangle FCG$, since each is the complement of ACB; $\therefore \triangle ABC \cong \triangle CGF$.

Rotate the $\triangle ABC$ about A through a minus r't $\measuredangle$; this brings B to B'. Likewise rotate CGF about F through a + r't $\measuredangle$; this brings G to G'. The sum of the angles at $D =$ st' $\measuredangle$.

$\therefore G'D$ and DB' are in one straight.

Produce GB to meet this straight at H; then $BC = GF = FG'$; and r't $\measuredangle G = \measuredangle GFG' = \measuredangle FGH$; $\therefore GFG'H$ equals square on BC.

Again, $BA = AB'$, and r't $\measuredangle B' = \measuredangle B'AB = \measuredangle ABH$; $\therefore ABHB$ is the sq' on AB.

$\therefore$ sq' of $AC =$ sq' of $AB +$ sq' of BC.

434. An altitude of a parallelogram is a perpendicular from a point in one side to the straight of the opposite side, which is then called the *base*.

435. Theorem. A parallelogram is equivalent to the rectangle of either altitude and its base.

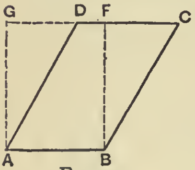


FIG. 201.

Proof. If CD, the side of the ‖g'm opposite the base AB, contains F, a vertex of the rectangle, then $ABFD \cong ABFD$, and $\triangle BCF \cong \triangle ADG$.

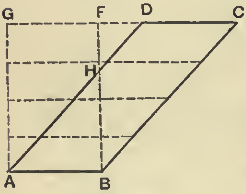


FIG. 202.

If the sides AD, BF intersect in H, then, by continued bisection, cut BF into equal parts each less than BH. Through these points draw straights ‖ to the base, so dividing the rectangle into congruent rectangles, each as above, equivalent to the corresponding parallelogram.

436. Corollary. All parallelograms having equal altitudes and equal bases are equivalent.

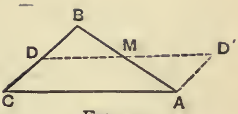


Fig. 203.

437. Theorem. A triangle is equivalent to the rectangle of its base and half-altitude.

Proof. Join the mid points D, M of the sides CB, BA of $\triangle\ ABC$, and produce $MD' = MD$. Then $\triangle\ AD'M \cong \triangle\ BDM$.

438. Corollary. Triangles of equal bases and altitudes are equivalent.

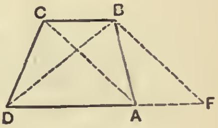


Fig. 204.

439. Theorem. A trapezoid is equivalent to a triangle of equal altitude, whose base is the sum of the parallel sides.

Proof. Join AC, BD. To DA produced draw $BF \parallel$ to CA.

$\triangle\ BCD = \triangle\ BCA \cong \triangle\ AFB$.

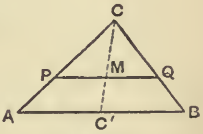


Fig. 205.

440. Theorem. The sect cut out, on a parallel to the base of a triangle by the sides, is bisected by the corresponding median.

Proof. Let M be the mid point of $PQ \parallel$ to AB. $\measuredangle\ PMC = \measuredangle\ QMC$; also trapezoid $AC'MP =$ trapezoid $C'BQM$; $\therefore\ AC'MCA = C'BCMC'$.

Were M not in CC', but on Q's side, then $AC'MCA > \triangle\ AC'C > C'BCMC'$.

441. Theorem. Sects joining intersections of the sides of a parallelogram with straights drawn parallel to the sides through a point on one diagonal, if they cut that diagonal, are parallel to the other.

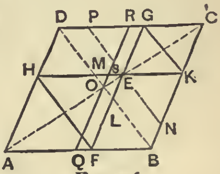


Fig. 206.

Proof. Through O draw $QR \parallel$ to BC, cutting HK in S. Since $DR = RC$, $\therefore\ MS = SE$, and $HM = EK = FB$; $\therefore\ HMBF$ is a $\parallel$g'm.

Again, since $MK = HE = DG$, $\therefore\ MKGD$ is a $\parallel$g'm.

442. Corollary. If through any point on a diagonal of a parallelogram straights be drawn parallel to the sides, the two

parallelograms, one on each side of this diagonal, will be equivalent.

For through E drawing $NP \parallel$ to BD, we get $\triangle\ FBL \cong \triangle\ HMD$, and $\triangle\ ENK \cong \triangle\ PEG$, and $\|$g'm $BNEL = \|$g'm $DMEP$.

443. Theorem. Any angle made with a side of a spherical triangle by joining its extremity to the circumcenter, equals half the angle-sum less the opposite angle of the triangle.

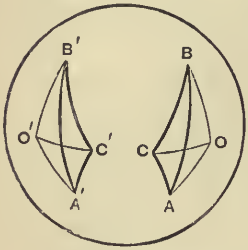


FIG. 207.

Proof. For $\measuredangle A + \measuredangle B + \measuredangle C = 2 \measuredangle OCA + 2 \measuredangle OCB \pm 2 \measuredangle OAB$

$\therefore \measuredangle OCA = \frac{1}{2}[\measuredangle A + \measuredangle B + \measuredangle C] - [\measuredangle OCB \pm \measuredangle OAB] = \frac{1}{2}[\measuredangle A + \measuredangle B + \measuredangle C] - \measuredangle B.$

444. Corollary. Symmetrical spherical triangles are equivalent.

For the three pairs of isosceles triangles formed by joining the vertices to the circumcenters, having respectively a side and two adjacent angles equal, are congruent.

445. Theorem. When three g-lines mutually intersect, the two triangles on opposite sides of any vertex are together equivalent to the lune with that vertical angle.

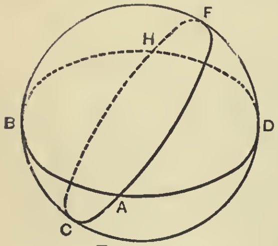


FIG. 208 *a*.

Proof. $\widehat{\triangle}\ ABC + \widehat{\triangle}\ ADF =$ lune $ABHCA$.

For $DF = BC$, having the common supplement CD; and $FA = CH$, having the common supplement HF; and $AD = BH$, having the common supplement HD; $\therefore\ \widehat{\triangle}\ ADF = \widehat{\triangle}\ BCH$; $\therefore\ \widehat{\triangle}\ ABC + \widehat{\triangle}\ ADF = \widehat{\triangle}\ ABC + \widehat{\triangle}\ BCH =$ lune $ABHCA$.

446. The *spherical excess*, e, of a spherical triangle is the excess of the sum of its angles over a straight angle. In general,

the spherical excess of a spherical polygon is the excess of the sum of its angles over as many straight angles as it has sides, less two.

447. Theorem. A spherical triangle is equivalent to a lune whose angle is half the triangle's spherical excess.

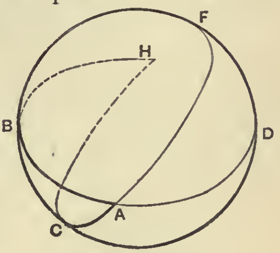


FIG. 208 *b*.

Proof. Produce the sides of the $\measuredangle$ ABC until they meet again two and two at D, F, and H. The $\measuredangle$ ABC now forms part of three lunes, whose angles are A, B, and C, respectively.

But, by 436, lune with $\measuredangle A = \widehat{\triangle}$ $ABC + \widehat{\triangle}\, ADF$.

Therefore the lunes whose angles are A, B, and C are together equal to a hemisphere plus twice $\widehat{\triangle}$ ABC. But a hemisphere is a lune whose angle is a straight angle; $\therefore$ $2\widehat{\triangle}$ $ABC =$ lune whose $\measuredangle$ is $[A + B + C -$ st. $\measuredangle] =$ lune whose $\measuredangle$ is e.

448. Corollary I. The sum of the $\measuredangle$s of a $\widehat{\triangle}$ is $>$ a st' $\measuredangle$ and $<$ 3 st' $\measuredangle$s.

449. Cor. II. Every $\measuredangle$ of a $\widehat{\triangle}$ is $> \frac{1}{2}e$.

450. Cor. III. A spherical polygon is equivalent to a lune whose angle is half the polygon's spherical excess.

451. Cor. IV. To construct a lune equivalent to any spherical polygon, add its angles, subtract $[n - 2]$ st' $\measuredangle$s, halve the remainder, and produce the arms of a half until they meet again.

EXERCISES ON BOOK III.

1. The joins of the centroid and vertices of a triangle trisect it.

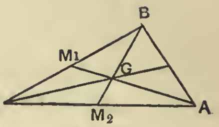


FIG. 267.

Proof. $\triangle\, ABM = \triangle\, MBC$,

$\triangle\, AGM = \triangle\, MGC$;

$\therefore \triangle\, ABG = \triangle\, GBC$.

2. Make a ‖g'm triple a given ‖g'm.
3. Make a $\triangle$ triple a given $\triangle$.
4. Make a symtra triple a given symtra.
5. Trisect a given symtra.
6. $(a + b)^2 = a^2 + 2ab + b^2$.
7. $(a - b)^2 = a^2 - 2ab + b^2$.
8. $(a + b)(a - b) = a^2 - b^2$.

10. On each side of a quad' describe a sq' outwardly. Of the four $\triangle$s made by joining their neighboring corners, two opposite equal the other two and equal the quad.

11. (Pappus.) Describe on two sides AB, AC of a $\triangle$ any ‖g'ms (both outwardly or both inwardly). Designate the cross of the sides opposite b and c by F. On the st' FA from a cut off $A'H = AF$. Construct a ‖g'm on a whose opposite side goes through H. It equals the sum of the other two.

12. If from an $\measuredangle\, \alpha$ we cut two $=$ $\triangle$s, one ⊹, the sq' of one of the $=$ sides of the ⊹ $\triangle$ equals the rectangle of the sides of the other $\triangle$ on the arms of the $\measuredangle\, \alpha$.

13. Transform a given $\triangle$ into an $=$ ⊹ $\triangle$.

14. Transform a given $\triangle$ into an $=$ regular $\triangle$.

15. If a vertex of a $\triangle$ moves on a $\perp$ to the opposite side, the difference of the squares of the other sides is constant.

16. The $\measuredangle$ bisectors of a rectangle make a sq', which is half the sq' on the difference of the sides of the rectangle.

17. The bisectors of the exterior $\measuredangle$s of a rectangle make a sq' which is half the square on the sum of the sides of the rectangle.

18. The sum of the squares made by the bisectors of the interior and

exterior ∡ˢ of a rectangle equals the square on its diagonal; their difference is double the rectangle.

19. If on the hypothenuse we lay off from each end its consecutive side, the sq' of the mid sect is double the rectangle of the others.

20. If in △ ABC, the foot of altitude from A be D, from C be F, then rectangles $BD \cdot a = BF \cdot c$.

(Hint. From ∡ B are two r't △s cut off. Turn one about the bisector of ∡ B.)

21. In a trapezoid, the sum of the squares on the diagonals equals the sum of the squares on the non-∥ sides plus twice the rectangle of the ∥ sides.

BOOK IV.

PROPORTION.

452. A greater magnitude is said to be a *multiple* of a lesser magnitude when the greater is the sum of a number of parts each equal to the lesser; that is, when the greater contains the lesser an exact number of times. The lesser is then called a *submultiple* of the greater.

453. Any multiple of any submultiple of a magnitude is called a *fraction* of that magnitude.

454. Two magnitudes of which neither is a fraction of the other are called *incommensurable;* for example 1 and $\sqrt{2}$.

455. That *definite numerical relation* of any magnitude to any magnitude of the same kind, in virtue of which the former is either a fraction of the latter or is greater than one and less than the other of two fractions of the latter differing by less than any given fraction however small, is called the *ratio* of the former to the latter.

456. If the first of two magnitudes is a fraction of the second, the ratio of the former to the latter is expressed by the numerical fraction whose denominator is the number indicating the submultiple of the second, and whose numerator is the number indicating the multiple of that submultiple.

Thus the ratio of a foot to 8 inches is 3/2.

457. The ratio of the first of two magnitudes to the second is said to be greater than a numerical fraction expressing the ratio, to the second, of any magnitude less than the first.

458. Two ratios are equal if no numerical fraction is greater than one and less than the other.

459. When the ratio of two magnitudes A and B, which

may be written A/B, equals that of the other two a and b, the four are said to form a proportion; which may be written $A/B = a/b$.

460. Theorem. If to every one of a series of magnitudes $A, B, C, \ldots$ there corresponds one of a second series $a, b, c, \ldots$ in such manner that,

I. If the magnitudes A and B are equal, so are also the corresponding magnitudes a and b; and,

II. The sum S of two magnitudes A and B corresponds to the sum s of the corresponding magnitudes a and b;

Then two magnitudes of the first series have the same ratio as the corresponding magnitudes of the second series.

Proof. 1. If B corresponds to b, and n is any integer, then nB corresponds to nb; for the sum of n equal parts B must [by II] correspond to the sum of n equal parts b.

2. Also the nth part of B corresponds to the nth part of b; for a magnitude which, taken n times gives B must correspond to that which taken n times gives b.

First Case. When A is a fraction of B.

Then $A = (n/d)B = n.(B/d)$.

Now [by 2] the magnitude B/d corresponds to b/d, and [by 1] the magnitude $n.(B/d)$ corresponds to $n(b/d)$.

Consequently $a = n.(b/d) = (n/d)b$.

Second Case. When A is no fraction of B.

Then if $A > (n/d)B$, $\therefore A/B > (n/d)B/B$, $\therefore A/B > n/d$.

But since $A > \dfrac{n}{d}B$, $\therefore$ [by II] $a > \dfrac{n}{d}b$, $\therefore a/b > n/d$.

461. Corollary I. If parallels cut two straights, the intercepts on one have the same ratio as the corresponding intercepts on the other.

For to sects $a, b, c, \ldots$ on one, the parallels give corresponding sects $a', b', c', \ldots$ on the other, such that if $a = b$, then $a' = b'$; and to the sum $a + b$ corresponds the sum $a' + b'$, etc.

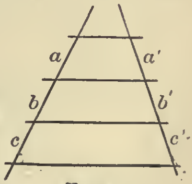


FIG. 209.

462. Cor. II. Parallelograms with an angle and a side in one equals to an angle and side in the other have the same ratio as their other sides.

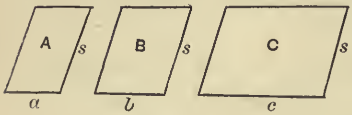


FIG. 210.

For this other side and the ‖g'm are then corresponding magnitudes, such that if of sides a, b, c, . . . and ‖g'ms A, B, C, . . . $a = b$, $\therefore A = B$, also to $a + b$ corresponds $A + B$.

463. Cor. III. In the same circle or in equal circles, angles at the center have the same ratio as their arcs.

For these angles A, B, C, . . . and arcs a, b, c, . . . so correspond that if $A = B$, then $a = b$; and to $A + B$ corresponds $a + b$.

464. Chords are not proportional to their arcs.

For if arcs A, B correspond to chords a, b, then arc $A + B$ does not correspond to a chord equal to $a + b$.

BOOK V.

SIMILARITY.

465. If from a point we draw rays to all the points of a given figure, and take on each of these rays another point, these latter points determine a second figure which we may call a *perspective* of the given figure. Two figures are called perspective when each point of one can be so paired with a point of the other that the joins of all the pairs concur in one point called the *center of perspective.* Two figures are called *projective* if they can be moved so as to be perspective.

Two figures are called *similar* when they can be so placed that on any straight whatever through one point sects from it to the perimeters of the figures have always the same ratio.

Figures are similar which, being projective, when made perspective have sects from the center of perspective to the pairs of points always in the same ratio.

466. The sect from the center of perspective to any point is called that point's *perspective sect.*

467. A point in the perimeter of one figure and a point in the perimeter of the other are said to correspond if their perspective sects are co-straight when the figures are in perspective position. Should the perimeters then have four points co-straight, of the two on one figure, that whose perspective sect is the lesser corresponds to that of the two on the other figure, whose perspective sect is the lesser.

468. The ratio of corresponding perspective sects is called the *ratio of similitude* of similar figures; the perspective center, their *center of similitude.*

469. The center of perspective is called *internal* when corresponding perspective sects lie on opposite sides of it; otherwise, *external.*

A symcenter is that special case of an internal center of perspective where the corresponding perspective sects are equal in magnitude.

470. Theorem. Any two circles are similar figures.

Proof. When concentric, their center is a center of perspective, and the ratio of corresponding perspective sects is constant, being the ratio of the radii.

471. Corollary. The intersection point of two straights is $\sim C$ for the arcs they intercept on circles with that point as center.

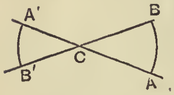


FIG. 211.

For the ratio of the radii gives a constant ratio of similitude.

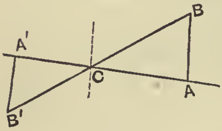


FIG. 212.

472. The intersection point of two straight, is $\sim C$ for the sects they cut out on any two parallels.

For a parallel through this point shows a constant ratio of similitude.

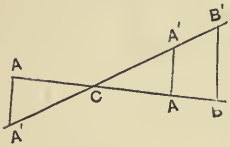


FIG. 213.

473. Theorem. If points on two straights are made corresponding which end sects from their intersection having the same ratio, the straights through corresponding points [projection-straights] are parallel.

Proof. By hypothesis $CA/CA' = CB/CB'$; $\therefore CA/CB = CA'/CB'$. But the parallel to AA' drawn through B gives $CA/CB = CA'/CB''$.

$\therefore B''$ coincides with B'.

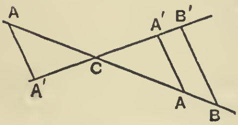


FIG. 214.

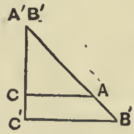


FIG. 215.

474. Theorem. Similar sects have the ratio of similitude.

Proof. When in perspective position since $CA/CB = CA'/CB'$, $\therefore$ the sects are parallel.

Now slide CBB' until B' comes on A' and BB' contains A. Thus A' becomes $\sim C$ for the parallels AC and BC'; $\therefore B'B/A'A = B'C'/A'C$.

475. Problem. To three given sects to find a fourth proportional.

Construction. On one arm of any $\measuredangle C$ cut off $CD = a$, and $DF = b$; on the other arm make $CH = c$. Join DH. Draw $FK \parallel$ to DH. $a/b = c/[HK]$.

476. We say that by a point P on the straight AB, but not on the sect AB, this sect is *divided externally;* and AP and BP are called *external segments* of the sect AB.

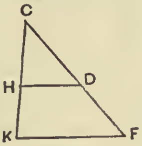


Fig. 216.

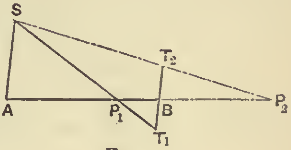


Fig. 217.

If the point P is on the sect AB, this is said to be *divided internally.*

477. Problem. To divide a given sect AB in a given ratio, AS/BT.

Construction. On parallels, from A and B take on opposite sides of the straight AB [or the same side] sects AS and BT. Join ST, cutting AB in P. Then $AP/PB = AS/BT$.

478. When a sect is divided internally and externally into segments having the same ratio, it is said to be divided *harmonically.*

479. Theorem. If a sect AB is divided harmonically by the points P and Q, the sect PQ will be divided harmonically by the points A and B.

Proof. Since $AP/BP = AQ/BQ$,

$$\therefore BP/AP = BQ/AQ;$$

$$\therefore BP/BQ = AP/AQ.$$

480. The points A, B, and P, Q, of which each pair divides harmonically the sect terminated by the other pair, are called four harmonic points or a *harmonic range.*

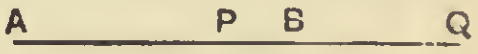


Fig. 218.

481. Problem. Given a point P on the straight AB, to determine the fourth harmonic point.

Construction. Through A and B draw parallels, and by a

straight through P, cut them in S and T. On straight BT take $BT' = BT$. The fourth point is on the straight ST'.

482. Corollary. With ABP only one point forms a harmonic range, for if S be any point without the straight in which is the harmonic range $APBQ$, and if through B we draw $BT \parallel$ to AS, meeting SP in T, SQ in T', then $BT = BT'$.

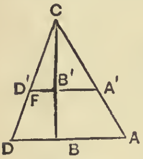


FIG. 219.

483. Theorem. If, of two parallels, those points which end proportional sects be joined, these projection-straights are concurrent.

Proof. Let the projection-straights AA' and BB' meet in C. Then let CD cut $A'B'$ in F. By hypothesis $AB/BD = A'B'/B'D'$. By 472 $AB/BD = A'B'/B'F$. Hence F coincides with D'.

484. Theorem. In any trapezoid the mid points of the parallel sides and the intersection point of the non-parallel sides and the intersection point of the diagonals form a harmonic range.

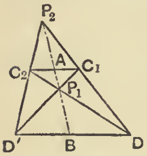


FIG. 220.

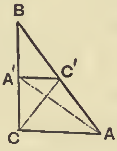


FIG. 221.

Proof. P divides the median AB in the ratio of the parallel sides.

485. Theorem. The medians of a triangle are concurrent in that trisection point of each remote from its vertex.

Proof. The sect joining the mid points of two sides of a triangle is $\parallel$ to and $\frac{1}{2}$ of the third side; $\therefore$ the intersection point of any two medians, since it divides each median in the ratio of these $\parallel$s, is that trisection point of each remote from its vertex.

486. The intersection point of its medians is called the triangle's *centroid.*

487. Theorem. The bisector of an exterior angle or interior angle of a triangle divides the opposite side externally or internally in the ratio of the other two sides of the triangle.

Proof. ABC any $\triangle$, BD the bisector of $\measuredangle$ at B. Draw

$AF \parallel BD$. Then of the two angles at B given equal by hypothesis, one equals the corresponding interior angle at F, and the other the corresponding alternate angle at A, $\therefore AB = BF$ [sides opposite equal $\measuredangle$s]. But $BF/BC = AD/DC$. [If parallels cut two straights, their intercepts are proportional.]

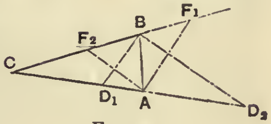


FIG. 222.

$$\therefore AB/BC = AD/DC.$$

488. Inverse. If one side of a triangle is divided internally or externally in the ratio of the other sides, the straight from the point of division to the opposite vertex bisects the interior or exterior angle.

489. Corollary. The bisectors of an interior and exterior angle at one vertex of a triangle divide the opposite side harmonically.

490. Theorem. Two triangles are similar if they have two angles respectively equal, or two sides proportional and the included angles equal, or two sides proportional and the angles opposite the greater equal, or their three sides proportional.

Proof. Put one angle upon its equal, and then the common vertex is $\sim C$.

The $\triangle$ with three sides proportional to those of a given $\triangle$ is $\cong$ to the $\triangle$ made by a straight $\parallel$ to one side of the given $\triangle$, and cutting off from a second side a sect equal to the corresponding side of the other $\triangle$.

491. Theorem. In a right triangle the altitude to the hypothenuse is a mean proportional between the segments of the hypothenuse, and each side is a mean proportional between the hypothenuse and its adjacent segment.

Proof. R't $\triangle ABC \sim \triangle ACD \sim \triangle CBD$.

492. Corollary. To find a mean proportional to two given sects, put a semicircle on their sum as diameter, and produce

to this semicircle the perpendicular erected at their common point.

493. Theorem. If four sects are proportional, the rectangle contained by the extremes is equivalent to the rectangle contained by the means.

Proof. Let the four sects a, b, c, d be proportional.

On a and on b construct rectangles with altitude c. On c

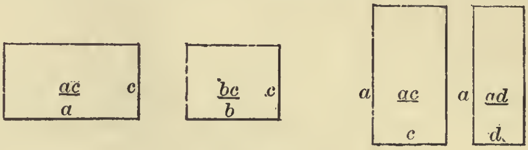


FIG. 223.

and on d construct rectangles of altitude a. Then $a/b = ac/bc$, and $c/d = ac/ad$.

[Rectangles of equal altitudes are to each other as their bases.] But by hypothesis, $a/b = c/d$.

$$\therefore ac/bc = ac/ad; \quad \therefore bc = ad.$$

494. Theorem. The rectangle of the segments into which a given point divides chords of a given circle is constant.

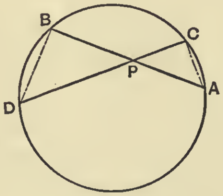


FIG. 224.

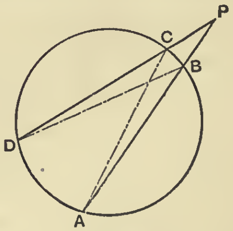


FIG. 225.

Hypothesis. Let chords AB and CD intersect in P.

Conclusion. Rectangle $AP.PB$ = rectangle $CP.PD$.

Proof. $\measuredangle PAC = \measuredangle PDB$ [inscribed angles on the same

arc], and $\measuredangle APC = \measuredangle BPD$; $\therefore \triangle APC \sim \triangle BPD$ [equiangular triangles].

$$\therefore AP/CP = PD/PB; \quad \therefore AP.PB = CP.PD.$$

495. Corollary. Let the point P be without the circle, and suppose DCP to revolve about P until C and D coincide: then the secant DCP becomes a tangent, and the rectangle $CP.PD$ becomes the square on PC. Therefore, if the point is without the circle, the rectangle is equivalent to the square of the tangent; if within, to the square on half the smallest chord.

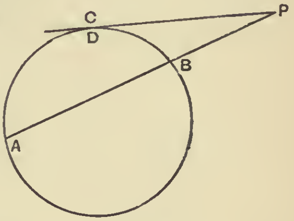


FIG. 226.

496. Theorem. If a triangle have two sides each equal to c, and from their intersection a sect d cut the third side into segments f and g, then $c^2 = d^2 + fg$.

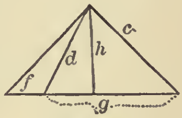


FIG. 227.

Proof. $$d^2 = h^2 + [\tfrac{1}{2}(f+g) - f]^2;$$

$$\therefore d^2 = h^2 + [\tfrac{1}{2}(g-f)]^2.$$

But $c^2 = h^2 + [\tfrac{1}{2}(g+f)]^2$.

497. Theorem. A point without a circle, and its chord of contact, divide harmonically any chord whose straight contains the point.

Proof. $AP.PB = c^2 = PQ^2 + CQ.QD = PQ^2 + AQ.QB$.

But $AP.PB = AP^2 + AP.AQ + AP.QB$; and $PQ^2 + AQ.QB = AP^2 + AP.AQ + AQ.PQ + AQ.QB$.

$\therefore AP.QB = AQ.PB$; $\therefore AP/PB = AQ/QB$.

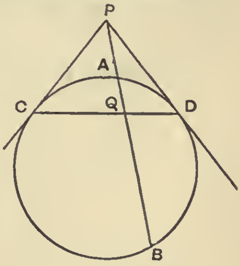


FIG. 228.

498. Theorem. The rectangles of opposite sides of a non-cyclic quadrilateral are together greater than the rectangle of its diagonals.

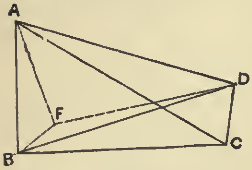


FIG. 229*b*.

Proof. Make $\measuredangle BAF = \measuredangle CAD$, and $\measuredangle ABF = \measuredangle ACD$.

Join FD.

Then $\triangle ABF \sim \triangle ACD$,

$$\therefore BA/AC = FA/AD.$$

But this shows (since $\measuredangle BAC = \measuredangle FAD$),

$$\triangle BAC \sim \triangle FAD.$$

From $\triangle ABF \sim \triangle ACD$,

$$\therefore AB . CD = BF . AC.$$

From $\triangle BAC \sim \triangle FAD$,

$$\therefore BC . AD = FD . AC.$$

$$\therefore AB . CD + BC . AD = BF . AC + FD . AC > BD . AC.$$

499. Corollary (Ptolemy). The rectangle of the diagonals of a cyclic quad' equals the sum of the rectangles of opposite sides.

For then F falls on BD.

EXERCISES ON BOOK V.

1. The joins of the vertex of one ∡ of a △ to the ends of that diameter of the circum ⊙ which is ⊥ to the opposite side are the bisectors of that ∡.

2. If 2 △s have a common base, they are as the segments into which the join of the vertices is divided by the common base.

3. The 3 external bisectors of the ∡s of a △ meet the sides co-straightly.

4. Given one side of a △, and the ratio of the other sides, find the path of its movable vertex.

5. The sect ∥ to one side of a quad' from the cross of 2 diagonals and bisected by the opposite side ends where?

6. If equiangular △s have a common vertex and second vertices co-st', so are the third vertices.

7. If c be the center-sect of the in- and circum-⊙s of a △, then

$$\frac{r}{R+c}+\frac{r}{R-c}=1.$$

8. The ⊙s on the sides of a △ as diameters cross on the sides of the △.

9. If a ⊙ be described on one of the ⊥ sides of a r't △ as diameter, the tangent at the p't where it divides the hypothenuse bisects the other ⊥ side.

10. The mid p'ts of concurrent chords are concyclic.

11. AA', BB', CC' are ∥ chords of a ⊙. Show △ ABC ∔ △ $A'B'C'$.

12. AB is trisected in C and D; CPD is a regular △; show that D is circumcenter of BPC, and AP the tangent at P to the circum-⊙.

13. Two △s on opposite sides of the same base have the ∡s opposite it supplemental. Show that the join of their supplemental ∡s is ∥ to the join of their orthocenters.

14. If the ⊥ projections of any vertex of a quad' on the other sides and diagonal of the quad' are co-straight, so are the like projections of any other vertex.

BOOK VI.

MENSURATION.

500. In practical science, every quantity is expressed by another of the same kind preceded by a number.

From our knowledge of the number and the quantity it multiplies, we get knowledge of the quantity to be expressed.

So in each kind of magnitude we select one convenient quantity as a standard or *unit*, to be known familiarly by us, and then to be used in expressing every other magnitude of the same kind.

The *measurement* of a magnitude consists in finding its ratio to its unit.

501. For sects, the unit is the centimeter [$^{\text{cm.}}$], which is the hundredth part of the sect between two marked points on a special bar of platinum at Paris, when the bar is at the temperature of melting ice.

The *length* of any sect is its ratio to the centimeter.

502. An accessible sect may be approximately measured by the direct application to it of a centimeter, or a sum of centimeters, such as the edge of a ruler suitably divided.

But because of incommensurability, even were our senses perfect, any direct measurement must be usually imperfect and merely approximate.

503. For the measurement of surfaces the standard is the square centimeter [$^{\text{cm.}^2}$], the square on the linear unit.

504. The *area* of any surface is its ratio to this square.

505. Theorem. The area of a rectangle equals the product of the length of its base by the length of its altitude.

Proof. If the altitude of the rectangle R is a, and its base b, then its ratio to a rectangle of altitude 1^{cm} and base b^{cm} is a; but the ratio of this rectangle to the square centimeter is b;

$$\therefore R = ab^{cm^2}.$$

506. Corollary. The area of a square is the second power of the number denoting the length of its side.

507. Cor. From the area of a square, to find the length of its side: extract the square root of its area.

508. To find the length of the other side, from the length of the hypothenuse and of one side of a right triangle, multiply the sum of the lengths by the difference, and extract the square root.

$$c^2 - a^2 = [c - a][c + a] = b^2.$$

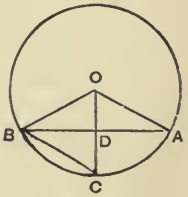


FIG. 229.

509. Given the chord of an arc and the radius of the circle, to find the chord of half the arc.

$$\overline{BC} = k' = \sqrt{\overline{BD}^2 + \overline{DC}^2} = \sqrt{[\tfrac{1}{2}k]^2 + \overline{DC}^2}.$$

But $$\overline{DC}^2 = [OC - OD]^2 = [r - OD]^2,$$

and $$OD = \sqrt{\overline{OB}^2 - \overline{BD}^2} = \sqrt{r^2 - \frac{k^2}{4}}.$$

$$\therefore k' = \sqrt{\frac{k^2}{4} + \left(r - \sqrt{r^2 - \frac{k^2}{4}}\right)^2} = \sqrt{2r^2 - 2r\sqrt{r^2 - \frac{k^2}{4}}}.$$

510. Since the angle at the center subtended by the side of a regular inscribed hexagon is one third a straight angle, and

so this side equals the radius, therefore the length of a side of a regular dodecagon inscribed in a circle whose radius is 1^{cm} is

$$k' = \sqrt{2-2\sqrt{1-\tfrac{1}{4}}} = 0.51763809+.$$

The length of one side of a regular inscribed polygon of 24 sides is

$$k'' = \sqrt{2-2\sqrt{1-\frac{(.51763809+)^2}{4}}} = 0.26105238+.$$

511. Given the radius of a circle and the side of a regular inscribed polygon, to find the side of a similar circumscribed polygon.

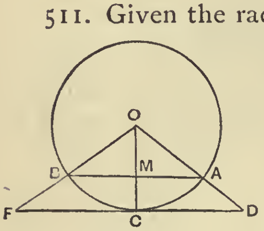


FIG. 230.

Suppose AB the given side k. Draw the tangent at the mid point C of the arc AB, and produce it both ways to the points D and F, where it meets the radii OA and OB produced. DF is the side required, t.

$$\triangle OFC \sim \triangle OBM,$$

$$\therefore\ OC/OM = CF/BM = t/k.$$

$$t = \frac{kr}{OM}.$$

But $$OM = \sqrt{\overline{OB}^2 - \overline{BM}^2} = \sqrt{r^2 - \tfrac{1}{4}k^2}.$$

$$\therefore\ t = \frac{kr}{\sqrt{r^2-\tfrac{1}{4}k^2}}.$$

512. Corollary. When $r = 1^{cm}$, the side of a regular circumscribed hexagon

$$t = \frac{1}{\sqrt{1-\tfrac{1}{4}}} = \frac{2}{\sqrt{3}}.$$

The side of a regular circumscribed dodecagon

$$t' = \frac{0.51763809}{\sqrt{1 - \frac{1}{4}(.51763809)^2}} = .535898 +.$$

513. Since no part of a circle can be congruent to any sect, so no part of a circle can be equivalent to any sect in accordance with our definition of equivalent magnitudes as such as can be cut into pieces congruent in pairs. Hence we assume:

[1] No arc is less than its chord.

[2] No arc is greater than the sum of the tangents at its extremities.

As a consequence of these paradoxical assumptions, an approximate value of a semicircle is given by the semiperimeter of every polygon inscribed or circumscribed. Moreover, the semicircle cannot be less than the inscribed semiperimeter nor greater than the circumscribed.

514. Calculating the length of a side in the regular inscribed and circumscribed polygons of 6, 12, 24, 48, 96, etc., sides, radius 1^{cm}, and in each case multiplying the length of one side by half the number of sides, we get the following table of semiperimeters:

n	$\frac{1}{2}nk_n$	$\frac{1}{2}nt_n$
6	3.0000000	3.4641016
12	3.1058285	3.2153903
24	3 1326286	3.1596599
48	3.1393502	3.1460862
96	3.1410319	3.1427146
192	3.1415424	3.1418730
384	3.1415576	3.1416627
768	3.1415838	3.1416101
1536	3.1415904	3.1415970
3072	3.1415921	3.1415937
6144	3.1415925	3.1415929
12288	3.1415926	3.1415927
393216 i.e., 6×2^{16}	3.1415926535	3.1415926537

515. Since a regular polygon of any number of sides, say 393216, is similar to any other regular polygon of that number of sides, therefore their sides have the same ratio as the radii of their circum-circles, or their in-circles. So 3.141592653 is not only an expression, exact to nine places of decimals, for the length of the semicircle whose radius is 1^{cm}, but also for the ratio of any semicircle to its radius.

516. This ratio of any circle to its diameter Euler designated by π.

The Bible [1 Kings vii. 23] gives for its value 3. The Egyptians twenty-two centuries before Christ gave $[4/3]^4 = 3.16$. Archimedes, from the perimeters of the regular inscribed and circumscribed polygons of 96 sides, placed it between $3\frac{10}{71}$ and $3\frac{1}{7}$. Ptolemy used $\pi = \frac{377}{120} = 3.1416$. The Hindoos gave $3927/1250 = 3.1416$.

Adriaan Anthoniszoon, father of Adriaan Metius [before 1589] gave $355/113 = 3.1415929$. Ludolf van Ceulen [1540–1610] gave $\pi = 3.14159265358979323846264338327950288$.

Lambert in 1761 demonstrated the irrationality of π. In June 1882 Professor Lindemann proved that π is a transcendental irrational, that is, π cannot be a root of a rational algebraic equation of any degree.

Hence the rectification of the circle is proved insoluble by compass and ruler.

CIRCULAR MEASURE OF AN ANGLE.

517. When its vertex is at the center of the circle,

$$\frac{\text{any } \measuredangle}{\text{st' } \measuredangle} = \frac{\text{its intercepted arc}}{\text{semicircle}} = \frac{\text{arc}}{r\pi}; \therefore \frac{\text{any } \measuredangle}{(1/\pi)\,\text{st' } \measuredangle} = \frac{\text{arc}}{r}.$$

So, adopting as unit angle $\frac{\text{st' } \measuredangle}{\pi}$, that is, the angle subtended at the center of every circle by the arc equal to its

radius, and hence called a *radian*, then *the ratio of any angle to the radian equals the ratio of its arc to the radius.*

If u denote the number of radians in an angle, and l its intercepted arc, then $u = l/r$.

The quotient *arc/radius*, or u, is called the *circular measure of an angle.*

518. Since a triangle is half the rectangle of either of its sides and the altitude to that side, therefore the area of a triangle is half the product of the length of a side by the length of its altitude.

519. Theorem. In any triangle, the square on a side opposite any acute angle is less than the sum of the squares on the other two sides by twice the rectangle contained by either of those sides and a sect from the foot of that side's altitude to the vertex of the acute angle.

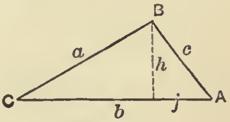


FIG. 231.

Proof. Let a, b, c denote the lengths of the sides, and h denote b's altitude, and j the sect from its foot to the acute angle A.

$$a^2 - h^2 = (b - j)^2 = b^2 - 2bj + j^2 = b^2 - 2bj + c^2 - h^2.$$

$$\therefore\ a^2 = b^2 - 2bj + c^2.$$

520. (Heron.) If $\triangle$ denote the area of any triangle and $s = \frac{1}{2}[a + b + c)$, then $\triangle = \sqrt{s[s-a][s-b][s-c]}$.

Proof. $j = \dfrac{b^2 + c^2 - a^2}{2b}$.

$$\therefore\ h^2 = c^2 - j^2 = c^2 - \frac{(b^2 + c^2 - a^2)^2}{4b^2},$$

$$\therefore\ 4h^2b^2 = 4b^2c^2 - [b^2 + c^2 - a^2)^2,$$

$$\therefore\ 2hb = \sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2},$$

$$\therefore\ 4\triangle = \sqrt{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)},$$

$$\triangle = \tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+b-c)(a-b+c)}.$$

521. The area of a regular polygon is half the product of its perimeter by the radius of the inscribed circle.

For sects from the center to the vertices divide the polygon into congruent isosceles triangles whose altitude is the radius, r, of the inscribed circle, and the sum of whose bases is the perimeter, p, of the polygon. $\therefore\ N = ap/2$.

522. The area of any circle $\odot = r^2\pi$.

For if a regular polygon of 393216 sides be circumscribed about the circle its area is $\frac{1}{2}rp$.

But $\frac{1}{2}p$ is $r\pi$; therefore its area is $r^2\pi$.

EXERCISES ON BOOK VI.

1. In a regular triangle the side (b) $= \frac{1}{3}$ perimeter (p) $= \sqrt{3}$ circum-radius (R) $= 2\sqrt{3}$ in-radius (r) $= \frac{2\sqrt{3}}{3}$ altitude (h) $= \frac{2}{3}\sqrt{3\sqrt{3}}$ area ($\triangledown$).

2. The area of a tangent-polygon (circum-polygon) is half perimeter by in-radius ($\frac{1}{2}pr$).

3. How many times greater does a quad' become if we magnify it until a diagonal is tripled?

4. Lengthening through A the side b of a $\triangle$ by c and c by b, they become diagonals of a symtra which is to the $\triangle$ as $(b + c)^2$ to bc.

5. One vertex of a ‖g'm and the mid points of the other two sides determine a $\triangle$. What is its ratio to the ‖g'm?

6. The squares of chords from the same point are as their $\perp$ projections on the diameter from that p't.

7. Make a sq' equal to $\frac{1}{3}$ a given sq'.

8. Make a sq' $= \frac{3}{5}$ a given sq'.

9. If from an $\measuredangle$ or from supplemental $\measuredangle$s we cut $+\triangle$s, they are as the sq's on one of the $=$ sides.

10. Trisect a $+\triangle$ by ‖s.

11. Their cross divides the non-‖ sides of a trapezoid externally, the diagonals internally, in the ratio of the ‖ sides.

12. (Circle of Apollonius.) If a sect is cut in a given ratio, and the interior and exterior points of division are taken as ends of a diameter, this circle contains the vertices of all triangles on the given sect whose other two sides have the given ratio.

13. If AD and BE are altitudes of $\triangle$ ABC, then $a/b = \frac{1}{AD} \Big/ \frac{1}{BE}$.

MISCELLANEOUS EXERCISES ON THE FIRST SIX BOOKS.

1. Describe a ⊙ having center in a given st' and containing two given points.

2. A ⊙ may be described which shall contain two p'ts, and have $r = a$; $(a > \frac{1}{2}AB)$.

3. $a + b + c > 2a$.

4. $a + b + c < 2a + 2b$.

5. If the sides of a regular polygon be produced to meet, their intersection points are the vertices of a similar polygon.

6. Trisect a st' ∡.

7. From a ⍿ △, cut a symtra with three sides =.

Hint. Join extremities of the two equal angle-bisectors.

8. Two external ∡ bi's of a ⍿ △ are ∥ to a side.

9. A median, a', is >, =, < a, according as ∡ A is acute, r't, or obtuse.

10. △s having a r't ∡ common, and = hy's, have mid's of hy's on a quadrant.

11. The angles made by productions of the sides of a reg' pentagon are together a st' ∡.

12. The angles made by productions of the sides of a reg' hex' are together a perigon.

13. Any two ∥g'ms on two sides of a △ are together = to a ∥g'm on the third side, whose consecutive side is = and ∥ to the sect joining the intersection of two sides produced of the other ∥g'ms to their common vertex.

14. The squares on the sides of a △ are together triple the squares on the sects joining the vertices to the centroid.

15. Triple the squares of the sides of a △ is quadruple the squares on the medians.

16. The sum of the sides of a △ is greater than the sum of its medians.

17. From the vertices, equal sects taken in order on the sides of a sq' give the vertices of a sq'.

18. With a vertex on a vertex, inscribe in a sq' a reg' △.

19. ∥g'ms inscribed in a ∥g'm have common sC.

20. If either diag' of a ∥g'm be = to a side, the other diag' > any side.

21. Sects from a point in a diag' of a ∥g'm to vertices give △s = in pairs.

22. One median of a trapezoid bisects it.

23. Sects from any p't in a ∥g'm to its vertices bisect it.

24. Sects from the mid p't of a non-∥ side of a trapezoid to opposite vertices bisect it.

25. Medians of a quad' bisect.

26. The sum of sq's of diag's of a trap' = sq's of non-∥ sides + two rect' of ∥ sides.

27. Draw a chord bisected by a given p't within a given ⊙.

28. Any chords which intersect on a diam', and make = ∡s with it, are =.

29. Describe a ⊙ with given *r*, center in given st', and tan' to another given st'.

30. The opposite sides of a circum-quad' subtend suppl' ∡s at the center.

31. *HD* produced to circum-⊙ is doubled.

32. In an inscribed even polygon, non-consecutive angles make half the angle-sum.

33. On a given sect as chord describe a segment which will contain a given ∡.

34. Find a curvilinear figure equivalent to a regular even polygon.

35. In a regular even polygon, any vertex and the center are co-st' with another vertex.

36. ∥ chords are sides of a symtra.

37. The sum of the squares of the segments of two ⊥ chords = d^2.

38. The ⊥ projections of opposite p'ts of a ⊙ on any st' are on a concentric ⊙.

39. If through a p't on a common chord pass two chords, their four extremities are concyclic.

40. $k_5^2 = k_6^2 + k_{10}^2$.

41. $t_3 : d :: d : t_6$.

42. $OA' + OB' + OC' = R + r$.

43. Any rectangle is half the rectangle of the diagonals of squares on its sides.

44. If two = chords intersect they make equal segments [they are diag's of a symtra].

45. A ∥ through the center is ½ perimeter of a circum-symtra.

46. $b : c :: \perp$ fr. A' on $c : \perp$ fr. A' on b.

47. The sum of the diag's of a quad' is less than the sum of any other four sects from a p't to the vertices.

48. Through a given p't draw a st' on which ⊥s from two given p'ts shall be =.

49. The ∡ bi's of a ∥g'm make a rectangle.

50. From the r't ∡, the median and altitude of the r't △ contain ∡ = dif' of the acute angles.

51. An angle-bi' and median contain ∡ = to dif' of other ∡s of the △.

52. If of the four △s into which the diag's divide a quad', two opposite are =, it is a trap'.

53. If two circles cut, the intercepts on any two ∥s through the points of section are =.

54. Chords all drawn from a p't on a ⊙ have their mid p'ts concyclic.

55. If from one common p't of two equal intersecting ⊙s as center a ⊙ be drawn, two of the points in which it cuts them, and their other common p't, are co-st'.

56. If two = ⊙s cut, the part of a st' through a common p't intercepted between them is bi'd by the ⊙ on their common chord as diameter.

57. If two ⊙s are tangent, two st's through the p't of contact intercept arcs whose chords are ∥.

58. If two ⊙s touch externally, and ∥ d's be drawn, a st' joining their extremities will contain the p't of contact.

59. In a st' through the center determine a p't from which a tan' shall be = d.

60. The ∡ made by tan's from a p't to a ⊙ is double the ∡ of chord of contact and diam' through a p't of contact.

61. Through a given p't to draw a st' which shall make equal ∡s with two given st's.

62. From two given p'ts to draw two = sects which shall meet on a given st'.

63. From two given p'ts on the same side of a given st' to draw two st's which shall cross on that st' and make = ∡s with it.

64. If a tan' be ∥ to a chord, the p't of contact will be the mid p't of the chord's arc.

65. Of st's drawn from two given p'ts to meet on a ⊙, the sum of those two will be least which make = ∡s with the tan' at the point of concourse.

66. If two ⊙s cut, and from either common p't diam's be drawn, their extremities and the other common p't are co-st'.

67. If a ⊙ be described on the *r* of another ⊙ as *d*, any sect from the common p't to the greater is bisected by the lesser.

68. The st's joining to the centre the intersections of a tan' with two ∥ tan's are ⊥.

69. St's from two p'ts in a ⊙ to a p't in a tan' make the greatest $\angle$ when drawn to the p't of contact.

70. If any chord be bisected by another, and produced to meet the tan's drawn at the extremities of this other, the parts between the tan's and the ⊙ are =.

71. If one chord bisect another, and tan's at the extremities of each be produced to meet, the join of their points of intersection is ∥ to the bisected chord.

72. If from the extremities of a diameter chords be drawn intersecting, two and two, on a ⊥ to that d', the joins of the extremities of the pairs are concurrent.

73. If from any p't in the base of $+\triangle$ st's making equal $\angle$s with the base be drawn to the sides, the △s formed by joining the intersections to the opposite vertices are =.

74. Which st' through a given p't within a given $\angle$ will cut off the least △?

75. The diag's of a trap' cross on a median.

76. A st' bisecting a side of a △ is cut harmonically by the three sides and a ∥ to the bisected side through the opposite vertex.

77. A st' from a vertex of a △ is cut harmonically by the opposite side, a median, and a ∥ to either of the other sides through the opposite vertex.

78. If from the ends of a side of a △ st's be drawn intersecting in the altitude to that side, the straights joining the points where they cross the other sides to the foot of that altitude make equal angles with it.

79. If from any $\angle$ of a rectangle a sect be drawn to a side, and a ⊥ to it from the adjacent $\angle$ of the quad' so formed, their rect' = the given rect'.

80. The two spherical tan's from a p't to a ⊙ are =.

81. If the g-lines joining the corresponding vertices of two $\widehat{\triangle}$'s concur, the crosses of opposite sides are collinear; and inversely.

82. If a spherical quad' is inscribed, and another circumscribed touching at the vertices of the first, the crosses of the opposite sides of these quad's are collinear.

[The crosses of the opposite sides of the inscribed are on the diagonals of the circumscribed.]

83. The crosses of the sides of an inscr' $\widehat{\triangle}$ with the spherical tan's at the opposite vertices are collinear.

84. If from the greater of two sides of a $\triangle$ a portion be cut off equal to the lesser, the join of the p't of section and the opposite $\measuredangle$ makes an $\measuredangle = \frac{1}{2}$ dif' of $\measuredangle$s adjacent to third side of $\triangle$.

85. Every $\odot$ passing through a given p't and centered in a given st', passes also through another fixed p't.

86. The rectangles of opposite sides are together double a cyclic quad' whose diag's are $\perp$.

87. If through the mid p't of any chord two chords be drawn, the joins of their extremities will cut off equal sects on the first chord.

88. In a r't $\triangle$ the dif' between the hy' and the sum of the other sides equals the *d* of the in-$\odot$.

89. If from the extremity of the radius of its circum-$\odot$ bisecting one side of a $\triangle$ a $\perp$ be drawn to the larger of the other two sides, one of the segments made is half the sum, the other half the difference, of these sides.

90. The center of $\odot$ touching semicircles described outwardly on the two sides of a r't $\triangle$ is the mid p't of the hypothenuse.

91. An angle-bi' of a $\triangle$ cuts the circum $\odot$ in the center of a $\odot$ containing the other two vertices and the in-center.

92. If from a vertex of a $\triangle$ inscribed in a $\odot$ st's be drawn $\parallel$ to the tangents at the extremities of the opposite side, they cut off $\sim\triangle$s.

93. The joins of the vertices and the points of contact of the in-$\odot$ of a $\triangle$ concur.

94. If from the ends of a side of a square $\odot$s be described, one with the side, the other with the diagonal, as radius, the lune formed equals the square.

95. If the diam' of a semi-$\odot$ be cut in pieces and on them semi-$\odot$s be described, these together equal the given semicircle.

96. In a given st' determine the p't at which st's from two given p'ts on the same side of it will contain the greatest $\measuredangle$.

97. If the rectangles of the segments of two intersecting sects are equal, their extremities are concyclic.

98. If two altitudes are equal, is the $\triangle$ isosceles?

99. If two medians are equal, is the $\triangle$ isosceles?

100. $\triangle\, ABC \sim \triangle\, A'B'C'$ [where A' bisects a; B', b; C', c.]

BOOK VII.

MODERN GEOMETRY.

CHAPTER I.

TRANSVERSALS.

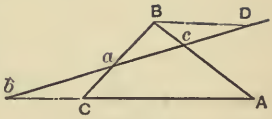


FIG. 232.

522. [Menelaus.] If the sides of the triangle ABC, or the sides produced, be cut by any transversal in the points a, b, c, respectively, then

$$[Ab/bC][Ca/aB][Bc/cA] = 1.$$

Inversely, given this relation, the points a, b, c will be co-straight.

Proof. Draw $BD \parallel$ to AC, and meeting the transversal in D:

then $\quad Bc/cA = BD/Ab, \quad$ and $\quad Ca/aB = bC/BD$;

therefore

$$[Ca/aB][Bc/cA] = [bC/BD][BD/Ab] = bC/Ab = 1/[Ab/bC].$$

Inversely, if the straight ab meet AB in c', then by Menelaus,

$$[Ab/bC][Ca/aB][Bc'/c'A] = 1.$$

But by hypothesis,

$$[Ab/bC][Ca/aB][Bc/cA] = 1.$$

Therefore c and c' coincide.

523. Corollary. If a traversal intersects the sides AB, BC, CD, etc., of any polygon in the points a, b, c, etc., in order, then

$$[Aa/aB][Bb/bC][Cc/cD][Dd/dE] \ldots \text{etc.} = 1.$$

Proof. Divide the polygon into triangles by straights through one vertex, apply Menelaus to each triangle, and combine the results.

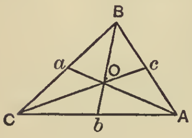


FIG. 233.

524. [Ceva.] If the sides of triangle ABC are cut by AO, BO, CO in a, b, c, then

$$[Ab/bC][Ca/aB][Bc/cA] = 1.$$

Inversely, given this relation, the straights Aa, Bb, Cc will be concurrent.

Proof. By the transversal Bb to the $\triangle AaC$ we have [Menelaus]

$$[Ab/bC][CB/Ba][aO/OA] = 1;$$

and by the transversal Cc to the AaB,

$$[Bc/cA][AO/Oa][aC/CB] = 1.$$

Multiply these equations together.

Inverse as in Menelaus.

525. Corollary. If transversals through O from the vertices of any odd polygon meet the sides AB, BC, CD, etc., in the points a, b, c, etc., in order, then

$$[Aa/aB][Bb/bC][Cc/cD][Dd/dE] \ldots \text{etc.} = 1.$$

526. Theorem. If any transversal cuts the sides of a triangle and their three intersectors AO, BO, CO, in the points A', B', C', a', b', c', respectively, then

$$[A'b'/b'C'][C'a'/a'B'][B'c'/c'A'] = 1.$$

Proof. Each side forms a triangle with its intersector and the transversal. Take the four remaining straights in succes-

sion for transversals to each triangle, applying Menelaus symmetrically, and combine the twelve equations.

527. Theorem. If the vertices of two triangles join concurrently, the pairs of corresponding sides intersect co-straightly, and inversely.

Proof. Take bc, ca, ab, transversals respectively to the triangles OBC, OCA, OAB; apply Menelaus, and the product of the three equations shows that P, Q, R lie on a transversal to ABC.

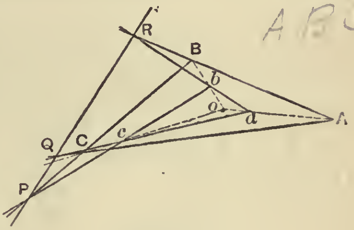


FIG. 234.

528. Corollary. If the vertices of any two polygons join concurrently, the pairs of corresponding sides intersect co-straightly.

529. The straight on which the pairs of sides cross is called the *axis of perspective.*

530. The $\perp$ *projection* of a point on a sect is the foot of the perpendicular from the point to the straight of the sect.

531. The $\perp$ projection of a sect on a straight is the piece between the perpendiculars dropped upon the straight from the ends of the sect.

532. Theorem. The $\perp$ projections on the sides of a triangle of any point on its circumcircle are co-straight.

[This straight is called the Simson's straight of the triangle with respect to the given point.]

Proof. Let O be any point on circumcircle of $\triangle ABC$. Join its $\perp$ projections GF, GH. Join OA, OC. Since $\measuredangle OGC$ and $\measuredangle OHC$ are r't, therefore C, H, G, O are concyclic. Similarly, G, B, F, O are concyclic.

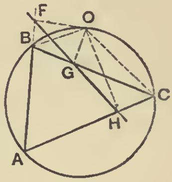


FIG. 235.

$\therefore \measuredangle OGF = \measuredangle OBF$, inscribed angles on same arc of circle $OGBF$.

But $\measuredangle OBF = \measuredangle OCA$, being supplemental to $\measuredangle OBA$.

$$\therefore \measuredangle OGH + \measuredangle OGF = \measuredangle OGH + \measuredangle OCH = \text{s't} \measuredangle.$$

533. Inverse. If the projections on the sides of a triangle of a point be co-straight, that point is on the triangle's circumcircle.

Proof. Let G, H, F, $\perp$ projections of O on a, b, c be co-straight. Since O, C, H, G are concyclic, $\therefore \measuredangle OCH = \measuredangle OGF$, being supplements of $\measuredangle OGH$.

But $\measuredangle OGF = \measuredangle OBF$, inscribed angles on same arc of circle $OGBF$.

$$\therefore \measuredangle OCA + \measuredangle OBA = \text{s't} \measuredangle.$$

$$\therefore O, C, A, B \text{ are concyclic.}$$

CHAPTER II.

HARMONIC RANGES AND PENCILS.

534. A system of co-straight points is called a *range*, of which the straight is the *bearer*.

535. A system of concurrent straights is called a *pencil*, of which the intersection point is the *vertex* or the *bearer*.

536. Straights all parallel form a pencil of parallels or a *parallel-pencil.*

537. Thus straights with equal perpendiculars from two given points form two pencils, one parallel to their join, and the other bisecting it.

538. A range and a pencil are called *perspective* when each point of the range lies on a straight of the pencil.

539. Two ranges are called perspective when their points lie in pairs on the straights of a pencil. The bearer of the pencil is called the *perspective-center.*

540. Two pencils are called perspective when their straights cross in pairs in the points of a range. The bearer of the range is called the projection axis.

541. Ranges and pencils are called *projective* if they can be put in perspective position.

542. If A, B be two points, and C, D, two co-straight with them, be so taken that $AC/BC = AD/DB$, then the points A, C, B, D form a harmonic range; C and D are *harmonic conjugates* with respect to A and B: AC, AB, AD are said to be in harmonic progression; and AB is said to be a harmonic mean between AC and AD.

Thus we have proved (479) that if C and D are harmonic conjugates with respect to A and B, then A and B are harmonic conjugates with respect to C and D.

543. If A, C, B, D form a harmonic range, and O be the mid point of AB, then $OA^2 = OB^2 = OC \cdot OD$.

For $AD/DB = AC/BC$.

$\therefore (AD + DB)/(AD - DB) = (AC + CB)/(AC - CB)$;

$\therefore 2OD/2OB = 2OB/2OC, \quad \therefore OB^2 = OC \cdot OD$.

544. Theorem. If four concurrent straights cut any transversal in a harmonic range, they will cut every transversal in a harmonic range.

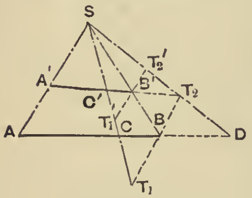


Fig. 236.

Proof. Through B and B' draw BT and $B'T \parallel$ to $AA'S$, and meeting SC in T_1, T_1' and SD in T_2, T_2'. Then since $BT_1 = BT_2$, $\therefore B'T_1' = B'T_2'$.

$A'C'B'D'$ is a harmonic range.

545. If A, C, B, D is a harmonic range, SA, SC, SB, SD is a *harmonic pencil*, and SC, SD are harmonic conjugates of SA, SB.

546. We have shown that the arms of any angle form with its internal bisector and its external bisector a harmonic pencil.

547. If, in a harmonic pencil, one element bisect the angle between two conjugates, then it is perpendicular to its conjugate.

548. If in a harmonic pencil one pair of conjugates be at right angles, then these are the internal and external bisectors of the angle between the other pair.

549. Theorem. If two harmonic ranges are taken, one in each of two straights, and if three of the four projection-straights are concurrent, then so are the four.

Proof. For the three concurrent projection-straights and the straight from their bearer through one of the fourth points form a harmonic pencil; so this latter straight contains also the other fourth point.

550. Corollary. If two corresponding points coincide in the cross of the two straights, then one projection-straight being free, the other three are always concurrent.

CHAPTER III.

PRINCIPLE OF DUALITY.

551. Not only the sect joining two points, but the whole straight, may be called the *join of the two points.*

552. The point common to two straights may be called the *cross of the two straights.*

553. In a pencil consisting of straights through one fixed point, any one of the straights may be called an element of the pencil, or a line on the fixed point or bearer.

In this sense, we say not only that points may lie on a straight, their bearer, but also that straights may lie on a point, their bearer, meaning that the straights pass through this point.

554. In most cases we can, when one figure is given, construct another, such that straights take the place of points in the first, and points the place of straights.

Thus from a definition or a theorem we can obtain another by interchanging *point* and *straight*, cross and join, *range* and *pencil*, or by similar interchanges.

555. A figure regarded as consisting of a system of straights crossing in points will thus give a figure which may be regarded as a system of points joined by straights; and in general with any figure coexists another having the same genesis from these elements, point and straight, but that these elements are interchanged.

Any descriptive theorem or theorem of position concerning

one, thus gives rise to a corresponding theorem concerning the other figure.

556. Figures or theorems related in this manner are called *reciprocal* figures or reciprocal theorems.

557. This correlation of point and straight is termed a *principle of duality.*

558. Each of two descriptive theorems so correlated is said to be *the dual* of the other; and it will be found that if any descriptive property is demonstrated, its dual also holds.

559. Since capitals mean points, and two fix a straight, their join; so small letters may denote straights, and two will fix a point, their cross.

Thus AB denotes the straight which is the join of the points A and B; while ab denotes the point which is the cross of the straights a and b.

560. In plane geometry to all points on a straight the reciprocal figure is all straights on a point.

561. A sect, AB, is that piece of a range containing the initial point A of the sect, its final point B, and all intermediate successive positions of the generating point.

562. The figure reciprocal to sect AB is $\measuredangle\, ab$, that piece of a pencil containing the initial straight a of the angle, its final straight b, and all intermediate positions of the generating straight.

RECIPROCAL THEOREMS.

563,. If two harmonic ranges are taken, one in each of two straights, and if three of the four projection-straights are concurrent, then so are the four.

563′. If two harmonic pencils are such that the three crosses of three pairs of corresponding straights are costraight, then this straight contains the cross of the fourth pair.

564_1. If two harmonic ranges are taken one in each of two straights, and two corresponding points coincide in the cross of the straights, then the other three projection-straights are concurrent.

564′. If two straights, one in each of two harmonic pencils, are coincident, then the three crosses of the other three pairs of straights are costraight.

CHAPTER IV.

COMPLETE QUADRILATERAL AND QUADRANGLE.

565_1. A system of four straights, no three concurrent, and their six crosses is called a *complete quadrilateral*, or *com-quad*.

565′. A system of four points, no three costraight, and their six joins is called a *quadrangle*.

566_1. The four straights are called the "sides" of the quadrilateral; and the six crosses, the vertices.

566′. The four points are called the *summits* of the quadrangle, and their six joins the "sides."

567_1. Two vertices which do not lie on the same "side" are called opposite vertices.

There are three pairs.

567′. Two sides which do not pass through the same summit are called opposite sides.

There are three pairs.

568_1. The three straights joining opposite vertices are called *diagonal straights*, and the triangle formed by the diagonal straights is called the *diagonal triangle* of the complete quadrilateral.

568′. The three crosses of opposite sides are called *diagonal points*, and the triangle determined by the diagonal points is called the *diagonal triangle* of the quadrangle.

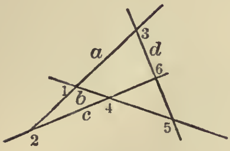


FIG. 237.

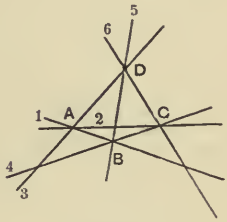


FIG. 238.

569₁. In this complete quadrilateral a, b, c, d, are the sides.

The vertices are 1, 2, 3, 4, 5, 6. 1 and 6 are opposite vertices. So are 2 and 5. Also 3 and 4.

569′. In this quadrangle A, B, C, D, are the summits.

The sides are 1, 2, 3, 4, 5, 6. 1 and 6 are opposite sides. So are 2 and 5. Also 3 and 4.

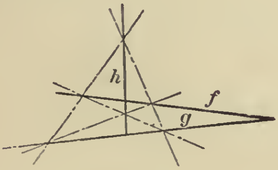


FIG. 239.

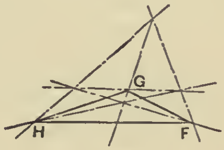


FIG. 240.

570_1. In the above complete quadrilateral, if f be the join of 1 and 6, g of 2 and 5, h of 3 and 4, then fgh is the diagonal triangle.

571_1. Theorem. In a complete quadrilateral each pair of opposite vertices forms with two of the angular points of the diagonal triangle a harmonic range.

570′. In the above quadrangle if F be the cross of 1 and 6, G of 2 and 5, H of 3 and 4, then FGH is the diagonal triangle.

571′. In a quadrangle, each pair of opposite sides forms with two of the sides of the diagonal triangle a harmonic pencil.

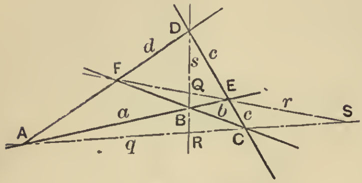


FIG. 241.

Proof. The range $q[ASC]$ is perspective with the range $r[ESF]$ to projection center B; $\therefore$ on a straight through B must lie the harmonic conjugates R and Q to S of these ranges.

But also $q[ASC]$ is perspective with $r[FSE]$ to projection center D; $\therefore$ also on a straight through D must lie R and Q. Hence they must lie on s, the join of B and D.

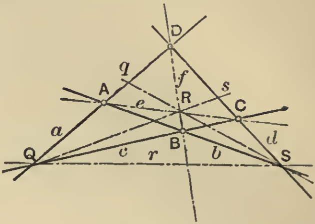


FIG. 242.

Proof. The pencil $Q[asc]$ is perspective with the pencil $R[esf]$ to projection axis b; on b must cross the harmonic conjugates r and q to s of these pencils.

But also $Q[asc]$ is projective to $R[fse]$ to projection axis d; $\therefore$ on d also must cross r and q. Hence they must go through S, the cross of b and d.

572. Problem. To draw a pair of tangents to a given circle from a given external point by means of a ruler only.

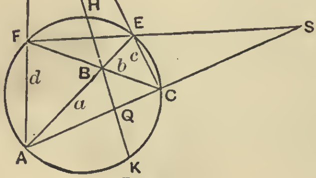
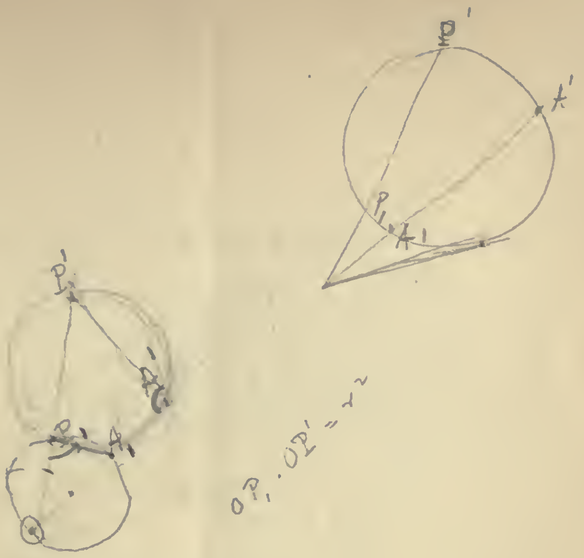


FIG. 243.

Construction. From the given point S, draw SCA, SEF, cutting the given circle in A, C and E, F. Join AE, CF, crossing at B. Join AF, CE, and produce to meet at D. The st' DB contains the chord of contact of S.

For we have proved in [497] that the chord of contact of S contains the harmonic conjugates R, Q of S on EF and AC, and we have just proved in [571_1] the opposite vertices BD of the complete quadrilateral *abcd* costraight with R, Q, these harmonic conjugates of S.



CHAPTER V.

INVERSION.

573. If on a ray from a fixed point O we take P_1 and P' such that the rectange $OP_1 . OP'$ equals the square on a fixed sect r, then the points P_1 and P' are termed each the *inverse* of the other with regard to O, the *center of inversion*, and r, the *radius of inversion.*

574. Any two points and their inverses are concyclic.

575. If P_1 moves on a certain line, then P' describes that line's inverse.

576. Theorem. The inverse of a circle through O is a straight perpendicular to the diameter through O. $\perp A'$

Proof. Let A_1 be the other end of the diameter through O, and P_1 any other point on the circle. Take P' and A' such that $OP_1 . OP' = OA_1 . OA' = r^2$. Then A_1, A', P_1, P' are concyclic, $\therefore \measuredangle\ OA'P' = \measuredangle\ OP_1A_1$, $\therefore P'$ is on the perpendicular to OA' through the fixed point A'.

577. Corollary. If the straight is tangent to the circle, the center of inversion is the other end of the diameter through the point of contact, and this diameter is the radius of inversion.

If the straight cuts the circle, either end of the diameter $\perp$

to it may be taken as the center of inversion, the radius of inversion being the sect from this to either point of section. Thus the circum-circle of an isosceles triangle can be inverted into the straight through the equal angles.

578. Theorem. The inverse of a circle not through O is another circle.

Proof. Draw OAB through the center of the given circle. Take the inverse points of A_1, B_1, P_1. Then $OP_1 . OP' = OA_1 . OA'$. $\therefore$ A_1, A', P_1, P' are concyclic, $\therefore$ $\measuredangle OA'P' = \measuredangle OP_1A_1$. In the same way $\measuredangle A'B'P' = \measuredangle B_1P_1P'$, $\therefore \measuredangle OA'P' + \measuredangle A'B'P' = \measuredangle OP_1A_1 + \measuredangle B_1P_1P' =$ r't $\measuredangle$ (since $\measuredangle A_1P_1B_1$ is r't), $\therefore$ $\measuredangle A'P'B'$ is r't. $\therefore$ P' is on $\odot$ with diamr' $A'B'$.

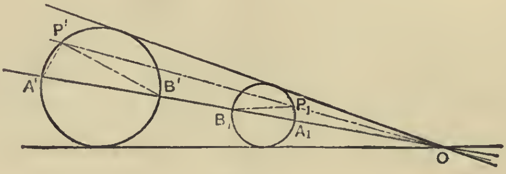


FIG. 245.

LINKAGE.

579. The Peaucellier Cell consists of a rhombus movably

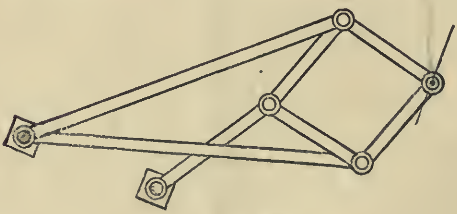


FIG. 246.

jointed, and two equal links movably pivoted at a fixed point, and at two opposite extremities of the rhombus.

TO DRAW A STRAIGHT LINE.

580. Take an extra link, and, while one extremity is on the fixed point of the cell, pivot the other extremity to a fixed point. Then pivot the first end to one of the free angles of

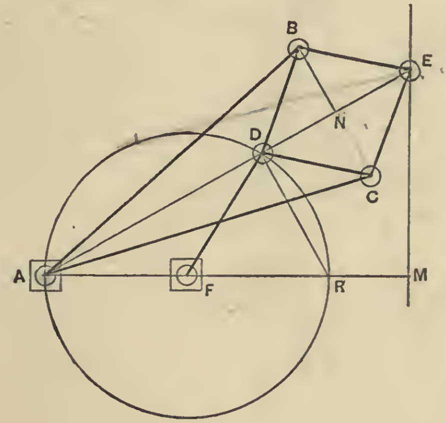


FIG. 247.

the rhombus. The opposite vertex of the rhombus will now describe a straight line, however the linkage be pushed or moved.

Proof. By the bar FD the point D is constrained to move on the circle ADR. A, D, E are always on the r't bi' of BC. Therefore, if $AE \cdot AD$ is constant, E moves on the straight EM. But $AE \cdot AD = [AN + NE][AN - NE] = AN^2 - NE^2 = [AN^2 + NB^2] - [NE^2 + NB^2] = AB^2 - BE^2 =$ a constant.

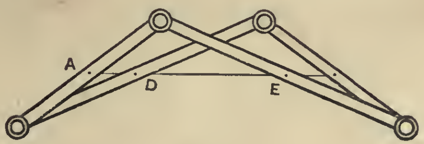


FIG. 248.

581. A linkage called Hart's Contraparallelogram is formed of four links $AB = CD$ sides, and $AC = BD$ diagonals of a symtra. The mid points O, P_1, P', O' are always costraight, and the rectangle $OP_1 \cdot OP'$, constant. For if M, H be the mid points of the symtra's $\|$ sides, then $OM = OH$ and $MP_1 = MP' = HP' = HP_1$. [The sect joining the mid points of two sides of a triangle is half the third]; so the relative position of the points OP_1P' is kept the same as in Peaucellier's Cell. So if O is fixed and P_1 describes any line, then P' must describe its inverse.

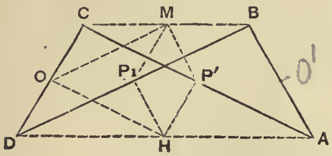


FIG. 249.

CHAPTER VI.

POLE AND POLAR WITH RESPECT TO A CIRCLE.

582. Theorem. If the cross of two tangents glides on a straight, their chord of contact rotates about a point; and inversely.

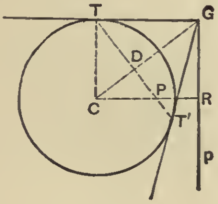


FIG. 250.

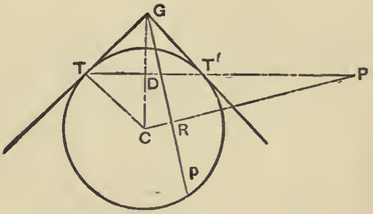


FIG. 251.

Proof. Draw $CR \perp$ to p, meeting TT' in P. Since CG is r't bi' of TT', $\therefore \measuredangle PDG$ is r't. $\therefore$ P, R, D, G are concyclic; $\therefore$ $CR . CP = CD . CG = CT^2$ [since TD is $\perp$ to hy' of r't $\triangle CTG$]. But CR and CT are fixed; $\therefore$ also CP. Inversely, draw $GR \perp$ to CP. Since, in the inverse, CP and CT are fixed, $\therefore$ so also is CR.

P is called the *pole* of p, and p the *polar* of P with respect to the given circle.

583. Since R and P are inverse points with respect to the center C, and radius CT, therefore the perpendicular to their

straight through either of two inverse points is the polar of the other, which is the pole of the perpendicular.

584. To get the pole of P with respect to a circle with center C, join PC cutting the circle in A and B, and through R, the harmonic conjugate of P with regard to AB, draw a perpendicular p.

585. Inversely, the pole of p with respect to a circle, center C, is the harmonic conjugate of the foot of the perpendicular from C on p with regard to the intersection points of the perpendicular with the circle.

586_1. If a straight passes through the pole of a second straight, so does the second straight pass through the pole of the first.

586′. If a point lie on the polar of a second point, so does the second point lie on the polar of the first.

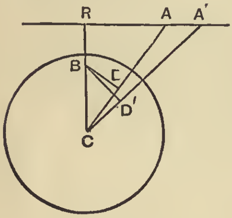


FIG. 252.

Proof. If A lie on AR the polar of B, then $CB \perp$ to AR. Draw $BD \perp$ to CA, then A, R, B, D are concyclic;

$\therefore CA \,.\, CD = CR \,.\, CB = r^2$; $\therefore BD$ is the polar of A.

587_1. Corollary. The join of the poles of two straights is the polar of their cross.

587′. The cross of the polars of two given points is the pole of their join.

588. Theorem. Any secant through a pole is cut harmonically by the circle, pole, and polar.

We have proved this in [497] if the pole be without the circle. If the pole be within the circle, we may substitute for it the intersection of its polar and the secant, since [586] the polar of that point contains the given pole.

589. A triangle of which each side is the polar of the opposite vertex with regard to a circle is said to be *self-conjugate* with respect to the circle.

590. The diagonal triangle of a quadrangle inscribed in a circle is self-conjugate.

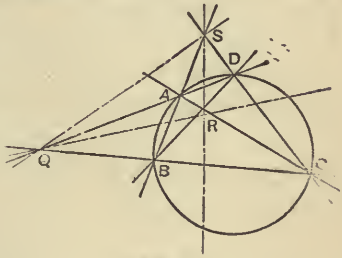


Fig. 253.

Proof. Ranges QAD, QBC are perspective from center S; ∴ the harmonic conjugates to Q are on a straight through S. But ranges QAD, QCB are perspective from center R; ∴ the harmonic conjugates to Q are on a straight through R. ∴ SR is the polar of Q. In the same way QR is the polar of S. ∴ QS is the polar of R.

591. Corollary. With ruler only, draw the polar of a given point, or find the pole of a given straight, with respect to a given circle.

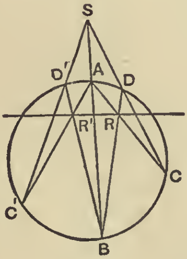


Fig. 254.

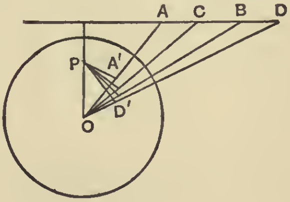


Fig. 255.

592₁. The polars of the four points of a harmonic range form a harmonic pencil.

592′. The poles of the four straights of a harmonic pencil form a harmonic range.

Proof. Let P be the pole of the straight $ACBD$ with respect to $\odot O$. Of A, C, B, D, the polars PA', PC', PB', PD', are $\perp$ to OA, OC, OB, OD. Thus the angles between the straights PA', PC', PB', PD' are respectively equal to the angles of the harmonic pencil OA, OC, OB, OD.

593. If with respect to a given fixed circle be taken the

pole of each straight, the polar of each point, of a figure F_1, we obtain a dual figure F'. This method is called *polarization* or *reciprocation*, and either of the figures is termed the *polar reciprocal* of the other, and any geometrical property of the one has its correlative for the other.

594_1. The pole of each straight through a point lies on the polar of this point.	594'. The polar of each point on a straight goes through the pole of this straight.
595_1. The join of the poles of two straights is the polar of their cross.	595'. The cross of the polars of two points is the pole of their join.
596_1. The poles of the straights of a pencil form a range whose bearer is the polar of the pencil's vertex.	596'. The polars of the points of a range form a pencil whose vertex is the pole of the range's bearer.

597. Thus in reciprocal polars correspond

in F_1	in F'
a join,	a cross,
a pencil,	a range,
parallels,	points co-straight with the center of reciprocation.
angle between two straights;	$\measuredangle$ [or its supplement] subtended by two points at center of reciprocation;

and *vice versa.*

598. A self-conjugate triangle is its own reciprocal polar.

599. The diagonal triangle of a cyclic quadrangle is also the diagonal triangle of the complete quadrilateral whose sides touch the circle at the vertices of the quadrangle.

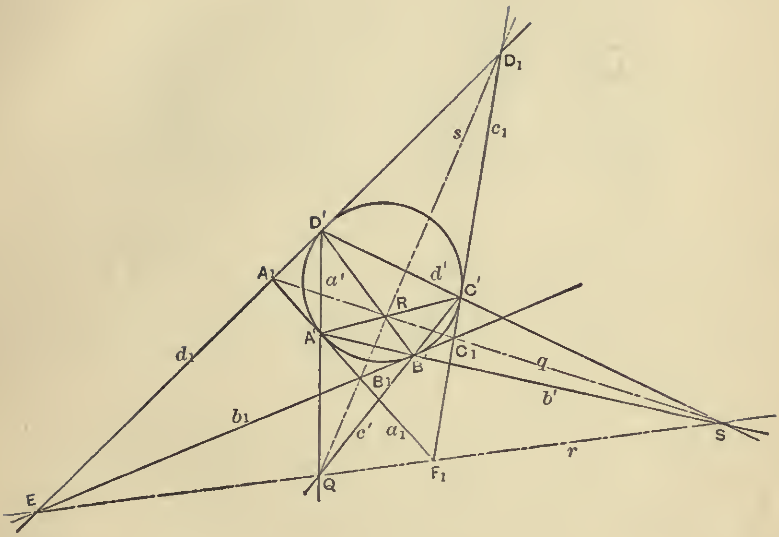


FIG. 256.

CHAPTER VII.

CROSS RATIO.

600. If in a range consisting of four points, A, B, C, D, we take A and B, called *conjugate points*, as the extremities of a sect, this is divided internally or externally by C; and distinguishing the "*step*" AC from CA as of opposite "sense," so that $AC = - CA$, the ratio AC/BC is never the same for two positions of C. The like is true of the positive or negative number AD/BD.

A B

$\frac{AC}{BC} \div \frac{AD}{BD}$

The ratio $[AC/BC]/[AD/BD]$ is called the *cross ratio* of the range, and is written $[\dot{A}\dot{B}CD]$.

601. Four elements may be arranged in twenty-four different ways:

$$\begin{array}{llll}
[\dot{A}\dot{B}CD], & [\dot{B}\dot{A}DC], & [\dot{C}\dot{D}AB], & [\dot{D}\dot{C}BA], \\
[\dot{A}\dot{B}DC], & [\dot{B}\dot{A}CD], & [\dot{D}\dot{C}AB], & [\dot{C}\dot{D}BA], \\
[\dot{A}\dot{C}BD], & [\dot{C}\dot{A}DB], & [\dot{B}\dot{D}AC], & [\dot{D}\dot{B}CA], \\
[\dot{A}\dot{C}DB], & [\dot{C}\dot{A}BD], & [\dot{D}\dot{B}AC], & [\dot{B}\dot{D}CA], \\
[\dot{A}\dot{D}BC], & [\dot{D}\dot{A}CB], & [\dot{B}\dot{C}AD], & [\dot{C}\dot{B}DA], \\
[\dot{A}\dot{D}CB], & [\dot{D}\dot{A}BC], & [\dot{C}\dot{B}AD], & [\dot{B}\dot{C}DA];
\end{array}$$

but four cross ratios in each of these six rows are equal, as may be readily proved by writing out any two in a row.

602. If in a cross ratio the two points belonging to one of the two groups be interchanged, the cross ratio changes to its reciprocal.

[Proved by writing out their values.]

Thus the ratio in the second row is reciprocal to that in the first, fourth to third, sixth to fifth.

603. If in a cross ratio the two middle letters be interchanged, the cross ratio changes to its complement.

$$[\dot{A}\dot{B}CD] = 1 - [\dot{A}\dot{C}BD].$$

For we have, taking account of sense or sign, $BC + CA + AB = 0$;

$\therefore\ BC.AD + CA.AD + AB.AD = 0$;

$\therefore\ BC.AD + CA.[BD + AB] + AB.[CD - CA] = 0$;

$\therefore\ BC.AD + CA.BD + AB.CD = 0$;

$\therefore\ 1 + [CA.BD]/[BC.AD] + [AB.CD]/[BC.AD] = 0$;

$\therefore\ 1 - [AB.CD]/[CB.AD] = [AC.BD]/[BC.AD]$;

$$\therefore\ 1 - \frac{AB}{CB}\Big/\frac{AD}{CD} = \frac{AC}{BC}\Big/\frac{AD}{BD};$$

$$\therefore\ 1 - [\dot{A}\dot{C}BD] = [\dot{A}\dot{B}CD].$$

604. By 603 $[\dot{A}\dot{D}CB] = 1 - [\dot{A}\dot{C}DB] =$ [by 602] $1 - 1/[\dot{A}\dot{C}BD] =$ [by 603] $1 - 1/[1 - (\dot{A}\dot{B}CD)]$.

Thus if the cross ratio $[\dot{A}\dot{B}CD] = \lambda$, then the six cross ratios derivable from these four co-straight points are λ, $\frac{1}{\lambda}$, $1 - \lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda - 1}{\lambda}$, $\frac{\lambda}{\lambda - 1}$.

605. Theorem. If S be a point without the range $ABCD$, and if through C a straight be drawn parallel to SD, meeting SA, SB in G, H, respectively, then $GC/HC = [\dot{A}\dot{B}CD]$.

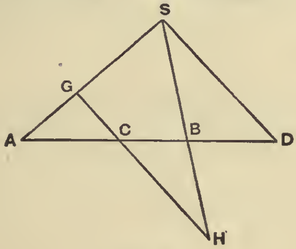


FIG. 257.

Proof. $GC/SD = CA/DA$. $SD/HC = DB/CB$.

$$\frac{GC}{SD} \cdot \frac{SD}{HC} = \frac{CA}{DA} \cdot \frac{DB}{CB} = \frac{CA}{DA} \Big/ \frac{CB}{DB};$$

$\therefore GC/HC = [\dot{C}\dot{D}AB] = [\dot{A}\dot{B}CD]$.

606. If two transversals meet the straights of the pencil $S\,[abcd]$ in A, B, C, D and in A', B', C', D', then $[\dot{A}\dot{B}CD] = [\dot{A}'\dot{B}'C'D']$.

Proof. Through C and C' draw GH and $G'H' \parallel$ to SD. Then $GC/HC = G'C'/H'C'$.

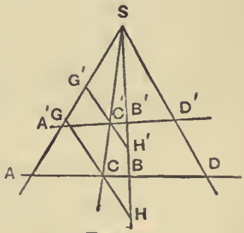


FIG. 258.

607. The cross ratio of the pencil $S\,[abcd]$ means the cross ratio of the four points $ABCD$ on any transversal, and is written $S\,[\dot{A}\dot{B}CD]$.

608. If two ranges or pencils have equal cross ratios they are said to be *equi-cross*.

609. Mutually equiangular pencils are equi-cross.

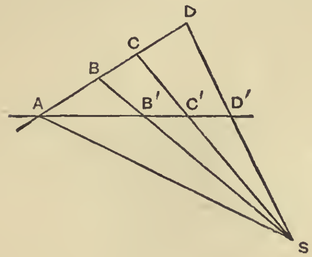


FIG. 259.

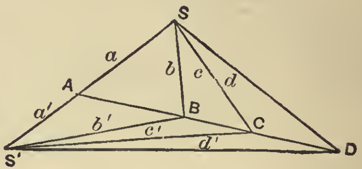


FIG. 260.

610_1. The joins of corresponding points of two equi-cross ranges which have two corresponding points coincident are concurrent.

610′. The crosses of corresponding straights of two equi-cross pencils which have two corresponding straights coincident are co-straight.

Proof. Let the join of the two crosses B and C cut the common straight in A, and cut d in D. Then is D also on d', since by hypothesis d' cuts ABC in a point D' such that $[\dot{A}\dot{B}CD] = [\dot{A}\dot{B}CD']$.

611. Corollary. Equi-cross ranges or pencils are projective.

612_1. Pencils whose straights pass through four fixed points on a circle, and whose vertices lie on the circle, are equi-cross.

Proof. The pencils are mutually equiangular.

612′. Ranges whose points lie on four fixed tangents to a circle and whose bearers are tangent to the circle are equi-cross.

Proof. Polarization from 612_1.

613_1. [Pascal.] In a cyclic hexagon the crosses of opposite sides are co-straight.

613′. [Brianchon.] In a circumscribed hexagon the joins of opposite vertices are concurrent.

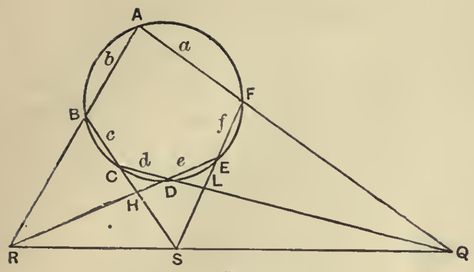


FIG. 261.

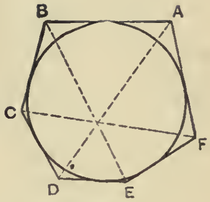


FIG. 262.

Proof. By 612_1, the pencils $B.ACDE$, and $F.ACDE$ are equi-cross; $\therefore$ $[\dot{R}\dot{H}DE] = [\dot{Q}\dot{C}DL]$; $\therefore$ by (610_1) RQ, HC, EL are concurrent.

614. If the figure formed by joining the six concyclic points by consecutive sects in any order be called a *hexagram*, there are 60, and Pascal holds for each.

615. Pascal holds for six points, three co-straight and also the other three.

INVOLUTION.

616. If a system of pairs of co-straight points AA', BB', CC', etc., be so situated with regard to a point O on the same straight that $OA \cdot OA' = OB \cdot OB' = OC \cdot OC'$, etc., they are said to be *in involution*. The point O is called the center, and AA', BB', CC', etc., are called *conjugate points of the involution.*

The points E, F, situated on the range, on opposite sides of C, such that $OE^2 = OF^2 = OA \cdot OA'$ are called the *double points of the involution.*

If straights be drawn from a point S outside the range to A, A', B, B', C, C', etc., they form a pencil in involution, and SE, SF are called the *double straights* of the pencil.

[Observing sense or sign, the double points and double straights are real only when conjugate points of the involution are on the same side of the center.]

617. Theorem. The two double points and any pair of conjugate points of an involution form a harmonic range.

For the sect between two double points is diameter of the circle with regard to which the conjugate points are inverses.

618. In a system of points or straights in involution the cross ratio of any four points or straights is equal to that of their conjugates.

619. If two pairs of conjugate straights of a pencil in involution be at right angles, then every pair of conjugate straights are at right angles.

RADICAL AXIS.

620. Corollary. The straights of a series of right angles at the same vertex form a system in involution.

621. Points from which tangents to two given circles are equal lie on a perpendicular to the center-sect which so divides it that the difference of the squares on the segments is equal to the difference of the squares on the radii of the two circles.

622. The bearer of the points from each of which tangents drawn to two given circles are equal is called the *radical axis* of the two circles.

623. If two circles intersect, their radical axis contains their common chord.

MILNE'S SYMMETRY THEORY OF MAXIMUM AND MINIMUM.

623_1. If a continuously varying magnitude, changing in accordance with some definite law, first increases until it attains a certain value and then decreases, that specific value, both preceded and followed by lesser values, is called a *maximum* value of the varying quantity.

Similarly, a value immediately preceded and followed by greater values of the variable is called a *minimum* value.

623_2. Just so the form of a geometrical figure, varying in a definite way, may approach symmetry, may attain symmetry, may immediately become unsymmetrical.

623_3. The positions which give the maximum and minimum

values of a continuously varying geometrical magnitude in any figure are positions of symmetry with regard to other parts of the figure which are fixed in position.

For example, the $\perp$ from a chord to its arc is a maximum when on the axis of symmetry of the figure.

Again, the perimeter of a triangle of fixed surface on a given base is a minimum when the $\triangle$ is ·|·.

623_4. Thus every varying geometrical magnitude may be considered to have two properties, one metrical, one positional or descriptive.

When the magnitude has a symmetrical position it has a maximum [minimum] value, and inversely. So we may reduce the problem of finding the maximum [min'] values of any varying geometrical quantity to the much simpler one of finding its positions of symmetry.

Ex. 1. The minimum [max'] sect between two $\odot$s is on their axis of ·|· [is on their center-st'].

Ex. 2. A and B are two fixed p'ts without a given $\odot$ O. Find a p't P on the $\odot$ such that $AP^2 + BP^2$ may be a minimum.

Bisect AB in C. Then $AP^2 + BP^2 = 2AC^2 + 2CP^2$.

Now AC is constant, $\therefore$ CP must be a minimum ; $\therefore$ the required p't is where CO cuts the $\odot$. [For max' take its other cross.]

Ex. 3. In Ex' 2 substitute a st' for the $\odot$.

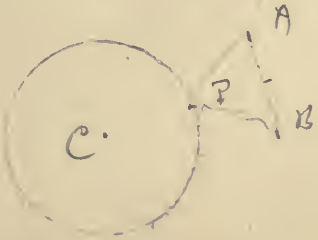
Ex. 4. Through a given p't draw a chord which shall cut off a minimum surface.

[The p't must be on the ·|· axis, $\therefore$ it bisects the chord.]

Ex. 5. Substitute in Ex' 4 two intersecting st's for the arc. [Same solution.]

Ex. 6. Through a given p't within a $\odot$ draw the minimum chord. [Same solution.]

Ex. 7. If two sides of a $\triangle$ be given in magnitude, the sur-



face is a maximum when each is ·|· axis for the st' of the other [when they are ⊥].

Ex. 8. To cut a sect so that the rectangle of the 2 pieces may be a maximum.

[The mid p't.]

[For the same p't, the sum of the sq's on the segments is min.]

Ex. 9. The p't within a sq' such that sq's on the ⊥s from it to the sides are together a minimum is its symcenter.

Ex. 10. If two sects cut ⊥, the sum of the rectangles of the segments of each is a maximum when they mutually bisect.

Ex. 11. In a given square inscribe the minimum sq.

Ex. 12. The p't within a △ such that the sum of the squares of its sects from the vertices is a minimum, is the centroid.

Ex. 13. Within a △ find a p't such that the sum of the squares on the ⊥s from it to the sides may be a minimum.

$$\left(\frac{p_1}{a}=\frac{p_2}{b}=\frac{p_3}{c}=\frac{2\triangle}{a^2+b^2+c^2}\right).$$

[For development of this theory see Milne's Companion to Problem Papers.]

EXERCISES ON BOOK VII.

1. The Simson-line of a p't bisects the join of that p't and the orthocenter of the △.

2. Circles described on any 3 chords from one p't of a ⊙ as diameters have their other 3 p'ts of intersection co-straight.

3. The circum-⊙s of the 4 △s formed by 4 intersecting straights concur.

4. The diameter of the in-⊙ of a r't △ and the hypothenuse together equal the other sides.

5. If *A*, *B*, *C*, *D* are concyclic, show that the Simson lines of *A*, *B*, *C*, *D* with respect to △s *BCD*, *CDA*, *DAB*, *ABC*, and the nine-point-circles of those △s, all pass through the same point.

6. The bisectors of the ∡s in a segment of a ⊙ form 2 pencils, whose bearers are the ends of the diameter bisecting the segment.

7. If from a p't within a △ ⊥s be drawn to the sides, then is the sum of the sq's of the three segments of the sides which have no common end point equal to the sum of the squares of the other 3.

[True when the p't is on or without the perimeter of the △.]

8. Inverse of the preceding.

9. If from a p't within a △ ⊥s be drawn to the sides, then is the sum of the 3 rectangles of side-segments having no common end p't each with its △-side equal the sum for the other 3, and equal half the sum of the squares of the sides of the △.

10. The 3 internal and 3 external bisectors of the ∡s of a △ meet the opposite sides in 6 p'ts, which are 3 by 3 in 4 st's.

11. If of 4 p'ts one is the orthocenter of the other 3, then every one is the orthocenter of the other 3.

12. *A*, *B*, *C* and their orthocenter *H* are the centers of the 4 ⊙s which touch *DEF*, the orthocentric △.

BOOK VIII.

RECENT GEOMETRY.

[The Lemoine-Brocard Geometry.]

CHAPTER I.

ANTI-PARALLELS, ISOGONALS, SYMMEDIANS.

624. Two mutually equiangular polygons are *co-sensal* when rays pivoted within them and containing the vertices of equal angles, rotate in the same sense to pass through the vertices of the consecutive equal angles.

625. If two transversals cross an $\angle$, so as to make with its arms two $\triangle$s equiangular, but not co-sensal, then either of these transversals is said to be *anti-parallel* to the other with regard to the angle.

Thus, if ABK, ACH be straights, and $\angle\ AKH = \angle\ ACB$, then KH anti-$\parallel$ to BC with respect to $\angle\ A$.

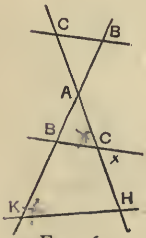


FIG. 263.

626. KB and HC are anti-$\parallel$ with regard to the $\angle$ of BC with KH.

627. B, C, H, K are concyclic; and inversely, if a quad' is cyclic, either opposite pair of its sides are anti-$\parallel$ with regard to the $\angle$ between the other pair.

628. Any st' $\parallel$ to the tan' to circum-$\odot$ of $\triangle$ ABC at A is anti-$\parallel$ to BC.

629. Anti-$\parallel$s to 2 sides of a $\triangle$ make the same $\angle$s with the 3d side.

Thus anti-$\parallel$s to the sides a and b of $\triangle\ ABC$ make each with c an $\angle = C$.

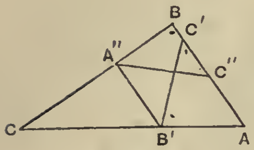


FIG. 264.

If the join of their 2 ends not in the 3d side is ∥ to it, they are =, since their 4 ends are then vertices of a symtra.

Inversely, if 2 anti-∥s are =, their 4 ends are vertices of a symtra.

In each case the center of the circle circumscribing the symtra is in the bisector of the ∡ between the anti-∥s.

630. The joins of the feet D, E, F of the altitudes of a △ ABC are anti-∥ to its sides. △ ABC is called the original triangle, and △ DEF is called the *orthocentric triangle*.

631. Given any anti-∥ $E'F$ to a within △ ABC; two anti-∥s, $F'D$, $D'E$, to b, c, can always be found within the △. [By drawing FD' ∥ to b and $E'D$ ∥ to c.]

632. Any two straights symmetrical with regard to an angle-bisector are called *isogonals* with reference to that angle.

633. If from two points, one on each of two isogonals with respect to a given angle, perpendiculars be drawn to its arms, then,

I. The rectangle of the perpendiculars to one arm equals that of those to the other;

II. The feet of the perpendiculars are concyclic;

III. The join of the feet of perpendiculars from the point on either isogonal is perpendicular to the other isogonal.

Proof. By ∼ △s,

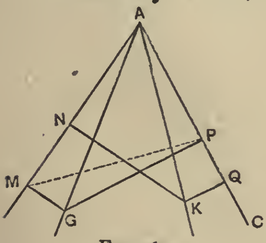


FIG. 265.

$$GM/GA = KQ/KA;$$
$$AG/GP = AK/KN;$$
$$\therefore GM/GP = KQ/KN;$$
$$\therefore GM \,.\, KN = GP \,.\, KQ.$$

II. By ∼ △s,

$$AM/AG = AQ/AK;$$
$$AG/AP = AK/AN;$$
$$\therefore AM/AP = AQ/AN;$$
$$\therefore AM \,.\, AN = AP \,.\, AQ,$$

∴ M, N, P, Q are concyclic.

III. Since $\measuredangle$s AMG, APG are r't, $\therefore$ $AMGP$ is cyclic;

$\therefore \measuredangle MAG = \measuredangle MPG$; but $\measuredangle GAM = \measuredangle KAQ$.

$\therefore \measuredangle MPG = \measuredangle KAQ$; but also $PG \perp$ to AQ;

$\therefore PM \perp$ to AK.

634. Inverse. If the rectangle of the $\perp$s from 2 given p'ts on one of the arms of a given $\measuredangle$ equal the rectangle of the $\perp$s on the other arm, the joins of the vertex and the p'ts are isogonal with respect to the $\measuredangle$.

635. If 3 st's through the vertices of a $\triangle$ concur, so do their isogonals.

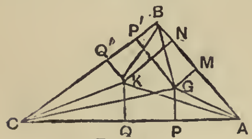


FIG. 266.

Proof. Let AG, BG be the isogonals of AK, BK. Then by 633,

$$p_2 p_2' = p_3 p_3',$$

and

$$p_1 p_1' = p_3 p'_3;$$

hence

$$p_2 p_2' = p_1 p_1',$$

$\therefore$ by 634, GC and KC are isogonals.

636. Two points so related to a triangle that the three joins of one to the vertices are isogonal to the joins of the other, are called *isogonal conjugates*.

637. Theorem. The six $\perp$ projections of 2 isogonal conjugates on the sides of the triangle are concyclic, and the center of this circle bisects their join.

Proof. By $\sim \triangle$s,

$$BQ'/BM = BK/BG = BN/BP';$$
$$\therefore BM \,.\, BN = BP' \,.\, BQ';$$

$\therefore M, N, P', Q'$ are concyclic.

And as the center of $\odot$ round them lies in the r't bi's of MN and $Q'P'$, it is the mid point of GK. Similarly, Q, P are on the same $\odot$.

638. Corollary I. The circumcenter and orthocenter of a △ are isog' conj's; ∴ a ⊙ passes through A', B', C', D, E, F, and the mid points of AH, BH, CH; with center, N, bisecting OH, and diameter R.

This is called the *nine-point circle.*

Its center is called the triangle's *medio-center.*

639. Corollary. $NI = \frac{1}{2}R - r.$

∴ the ninepoint-⊙ touches the in-⊙ and each ex-⊙ (Feuerbach's Theorem).

640. The isogonal conjugate to the centroid of a △ is called the *Lemoine point* of the △.

641. The isogonals to the medians of a △ are called its *symmedians.*

642. Since a median bisects all ‖s to its side of the △, ∴ by symmetry its symmedian bisects all anti-‖s to the side.

643. The Lemoine point bisects 3 anti-‖s, which are equal, since the two halves going to a side make with it ∡s each = to the opposite ∡ of the △. Thus the ends of any 2 of these are vertices of a rectangle.

644. The circle through the 6 p'ts in which anti-‖s to the sides of a △, through its Lemoine p't, meet the sides, is called the 2d Lemoine ⊙ of that △.

645. Since the sides of a △ and of its orthocentric △ are anti-‖, ∴ the sides of the orthocentric are bisected by the symmedians. ∴ join each angle of a △ to the mid point of that side of the orthocentric which ends in its arms; the joins concur in the Lemoine p't.

646. The ⊥s from the Lemoine p't to the sides of a △ are proportional to the sides.

From B' bisecting b, and K, the Lemoine p't, draw ⊥s. Then by ∼ △s, $B'P_1 \,.\, KP_1' = B'P_3 \,.\, KP_3'$. But since △ $ABB' =$ △ $B'BC$, $\therefore B'P_1 \,.\, a = B'P_3 \,.\, c$.

$\therefore KP_1'/a = KP_3'/c.$

647. If k_1, k_2, k_3 are $\perp$s from K on a, b, c, then

$$\frac{k_1}{a} = \frac{k_2}{b} = \frac{k_3}{c} = \frac{2\Delta}{a^2+b^2+c^2}.$$

648. [Grebe.] Describe sq's $APQB$, $BUVC$, $CXYA$ on the sides of $\triangle$ ABC [all externally or all internally], and let QP, XY meet in α; PQ, VU in β; UV, YX in γ; then αA, βB, γC concur in K.

649. [Mathieu.] The Lemoine point of a triangle is the center of perspective of that triangle and its polar triangle with respect to any circle.

Proof. With respect to circum-$\odot$ of $\triangle$ ABC, the pole of BC is P, of CA is Q, of AB is R. $LPM \parallel$ to QR is anti-$\parallel$ to BC. $\therefore PL = PB = PC = PM$; $\therefore AP$ is a symmedian of $\triangle$ ABC, $\therefore$ the Lemoine point is in AP.

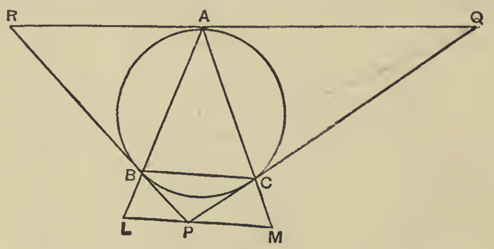


FIG. 268.

In same way, it is in BQ and CR; $\therefore$ it is $\wedge C$ of $\triangle$s ABC, PQR; and $\therefore$ of $\triangle$ ABC and any other of its polar $\triangle$s.

650. The joins of the points of contact of the in-$\odot$ of a $\triangle$ with its opposite vertices concur in the Lemoine p't of the $\triangle$ formed by joining the points of contact.

This is called the *Gergonne* point of the first $\triangle$.

651. [Schlömilch.] The three joins of the mid point of each side of a triangle, to the mid point of the corresponding altitude, concur in the Lemoine point.

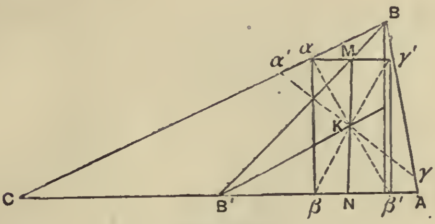


FIG. 269.

In $\triangle$ ABC let α, α' be p'ts in BC; β, β' in CA, γ, γ' in AB; such that $\alpha K \beta'$, $\beta K \gamma'$, $\gamma K \alpha'$ are the respective anti-$\|$s through K [the Lemoine p't], to AB, BC, CA.

$\therefore$ K is the mid p't of these sects and $\alpha\beta\beta'\gamma'$ is a rect'.

The median BB' [of $\triangle$ ABC] cuts $\alpha\gamma'$ in its mid p't M; and if MK meets AC in N, $KM = KN$; and MKN is $\|$ to $\alpha\beta$, and $\therefore$ to BE, the alt' from B. $\therefore$ $B'K$ is median of $\triangle$ $B'BE$ and $\therefore$ bisects BE.

652. [R. F. Davis.] If of six points a pair is in each side of a triangle and concyclic with each other such pair, then the six are concyclic.

Proof. $A\gamma_1 . A\gamma_2 = A\beta_1 . A\beta_2$; $\therefore$ A is on the radical axis of $\odot$s $\alpha_1\alpha_2\gamma_1\gamma_2$, $\alpha_1\alpha_2\beta_1\beta_2$.

But if these circles are distinct, yet intersect, their radical axis contains their common chord $\alpha_1\alpha_2$.

$\therefore$ They coincide.

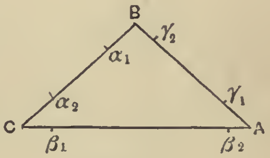


FIG. 270.

653. If $\alpha_1\beta_2$ anti-$\|$ to $\alpha_2\beta_1$, and
$\alpha_1\gamma_2$ anti-$\|$ to $\alpha_2\gamma_1$, and
$\beta_1\gamma_2$ anti-$\|$ to $\beta_2\gamma_1$, then the six points
α_1, α_2, β_1, β_2, γ_1, γ_2 are concyclic.

654. [Tucker.] The six ends of three equal anti-parallels in a triangle are concyclic.

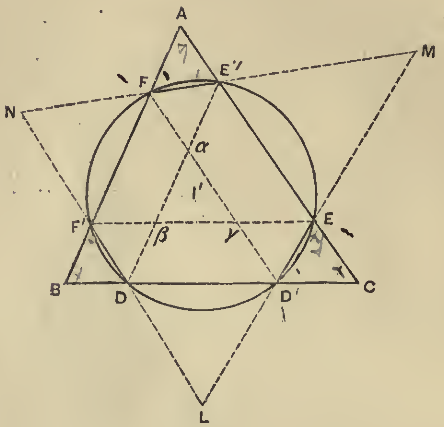


FIG. 271.

Proof. Let I' be the in-center of $\triangle LMN$.

Since $E'F$, $F'D$ make $\measuredangle\ E'FA = \measuredangle\ BF'D$, $\therefore\ \triangle FNF'$ is ·|·; $\therefore$ a bisector of $\measuredangle\ N$ is r't bi' of FF'; $\therefore$ the r't bi' of FF' passes through I'. Similarly, the r't bi' of DD' passes through I'. But since $E'F = D'E$, $\therefore\ D'F \parallel$ to AC, $\therefore\ D'$, F, F', D are concyclic; $\therefore\ I'$ is the center of a $\odot$ through D', F, F', D. Similarly, I' is center of $\odot$ through E', D, D', E.

655. The circles got by varying the size of the three equal anti-parallels are all called *Tucker's circles*.

656. Corollary. K is cost' with O and O' the circumcenters of $\triangle ABC$ and $\triangle \alpha\beta\gamma$; and I' bisects the join OO'. For, since $\alpha E'AF$ is a $\parallel$ g'm, $\therefore\ K$ is $\wedge\!\!\!\backslash$ C of $\triangle ABC$ and $\triangle \alpha\beta\gamma$.

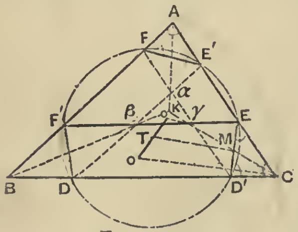


FIG. 272.

Also OC, $\perp$ to tan' at C, is $\perp$ to $D'E$ anti-$\parallel$ to c; and $\therefore\ O'\gamma$, $\parallel$ to OC, is $\perp$ to $D'E$; $\therefore$ through M (mid p't of γC and of $D'E$) a $\parallel$ to OC will bisect OO' in T and contain I'. Similarly for $E'F$; $\therefore\ T$ is I'.

657. Straights through the Lemoine point of a triangle parallel to its sides, are called the *Lemoine parallels* of that triangle.

658. The crosses of the Lemoine parallels and the sides of a triangle are concyclic.

For K being now the common vertex of 3 ‖g'ms, its joins to the vertices of the △ bisect the other 3 diagonals, joins of these crosses, which are thus 3 anti-‖s, and all equal, being non-‖ sides of 3 symtras.

659. This Tucker's circle through the ends of the Lemoine parallels is called the *First Lemoine Circle.*

660. The center of the First Lemoine Circle is the mid point of the sect KO. For $\triangle\alpha\beta\gamma$ has become the point K; so I', bisecting OO', now bisects OK.

660 (b). [H. M. Taylor.] The six ⊥ projections of the feet of its altitudes on the other sides of a △ are on a Tucker's ⊙, called its Taylor's ⊙.

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CHAPTER II.

THE BROCARD POINTS.

661. Problem. To draw a circle which shall pass through a given point and touch a given straight at a given point.

Construction. The $\perp$ to the st' at the given p't and the r't bi' of the join of the two given points cross in the center.

662. Problem. To find within a $\triangle$ a p't which its sides will contain after equal positive rotations about their vertices.

Construction. Describe a $\odot$ passing through one vertex, as C, and touching the opposite side c at the next counter-clockwise vertex, A.

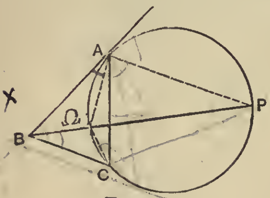

Fig. 273.

Draw the chord $AP \parallel$ to BC.

Join BP, cutting the $\odot$ in Ω.

Proof. Then [periphery $\measuredangle^s$ on the same arc ΩA] $\measuredangle\, AC\Omega = \measuredangle\, BA\Omega = \measuredangle\, AP\Omega = \measuredangle\, CB\Omega$ [its alternate $\measuredangle$].

Determination. Only one solution.

663. This p't Ω is called the *positive Brocard p't.* Its isogonal conjugate, the *negative Brocard p't* Ω', is given by substituting negative for positive in the preceding problem.

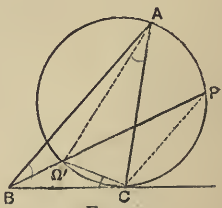

Fig. 274.

ixon p. 394 664. Ω and Ω' are isogonal conjugates.

665. The magnitude of the angle of rotation for Ω and for Ω' is designated by ω, and $\measuredangle + \omega$ is called the Brocard $\measuredangle$ of the $\triangle$.

666. The 3 ⊙s each passing through a vertex and touching the opposite side at the next counter-clockwise vertex concur in the positive Brocard p't.

[If next clockwise vertex, in Ω'.]

667. At A draw $AX \parallel$ to BC.

At C make $\measuredangle\ XCA = \measuredangle\ CBA$.

Then $\measuredangle\ CBX$ is the Brocard $\measuredangle$ of $\triangle\ ABC$.

668. From 667, the Brocard $\measuredangle$, $\measuredangle + \omega$, is the same for all $\sim \triangle$s.

669. In the construction 662 we may keep $\measuredangle\ B$ constant and increase $\measuredangle\ \omega$, by sliding $BC \parallel$ to itself until it touches the ⊙;

$\therefore$ if one $\measuredangle$ of a $\triangle$ is fixed, ω is greatest when the $\triangle$ is ·|·;

$\therefore$ an equilateral $\triangle$ has the greatest of all Brocard angles, which is $\frac{1}{3}$ r't $\measuredangle$.

670. The arcs $A\Omega C$, $A\Omega' B$, $B\Omega A$, $B\Omega' C$, $C\Omega B$, $C\Omega' A$ are called the *Brocard arcs* of the triangle.

671. [Brocard.] If O is circumcenter and K Lemoine of $\triangle\ ABC$; and if ⊙ on diameter OK cuts the Lemoine $\parallel$s to BC, CA, AB, in α, β, γ, respectively; then

I. $A\gamma$, $B\alpha$, $C\beta$ concur on the ⊙; and their cross is the positive Brocard point Ω.

II. $A\beta$, $B\gamma$, $C\alpha$ concur on the circle, and their cross is the negative Brocard point Ω'.

Proof. Let $EK\alpha F'$, $FK\beta D'$, $D\gamma KE'$ be the Lemoine $\parallel$s.

Then $\measuredangle\ O\alpha K$ is r't.

$\therefore$ $\alpha O \perp$ to BC bisects BC in A' [since O is circumcenter].

In same way βO, γO meet b, c in their mid points B', C'.

Let $B\alpha$, $C\beta$ cross in Ω;

then $\alpha A'/BA'$ = twice $\perp$ from K on BC/BC;

= twice $\perp$ from K on b/b [646];

$= \beta B'/CB'$.

$\therefore$ $\triangle\ \alpha BA' \sim \triangle\ \beta CB'$.

$\therefore$ $\measuredangle\ B\alpha A' = \measuredangle\ C\beta B' = \measuredangle\ O\beta\Omega$.

660

660

$\therefore$ Ω is concyclic with α, β, O, and $\therefore$ also with γ.

Similarly, $A\gamma$, $B\alpha$ concur on this $\odot$;

i.e., $A\gamma$, $B\alpha$, $C\beta$ concur in Ω on the $\odot$ whose diam' is OK.

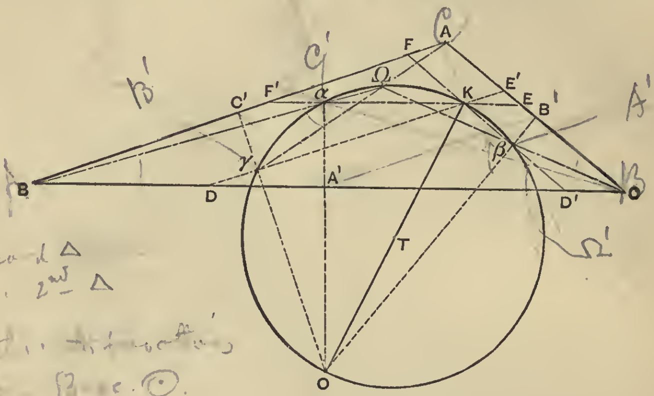

FIG. 275.

Similarly, $A\beta$, $B\gamma$, $C\alpha$ concur in a p't Ω' on the $\odot$.

Moreover, $\measuredangle \Omega BC = \measuredangle \Omega CA = \measuredangle \Omega AB$ [from $\sim$ $\triangle$s $\alpha BA'$, $\beta CB'$, $\gamma AC'$]. $\therefore$ Ω is the positive Brocard p't.

Similarly, Ω' is the negative Brocard p't.

672. The $\odot$ on the joint of the circumcenter and Lemoine p't of a $\triangle$ as diameter is called the Brocard $\odot$ of the $\triangle$, from the name of its discoverer.

673. The $\triangle$ whose vertices are the $\perp$ projections of the circumcenter on the Lemoine $\|$s is called Brocard's first $\triangle$.

674. The $\triangle$ whose vertices are the $\perp$ projections of the circumcenter on the symmedians is called Brocard's second $\triangle$.

675. In $\triangle ABC$ to inscribe a $\triangle$ $\sim$ to a given $\triangle$.

In AB take any p't D, and draw any sect DF to another side, as AC; and at the points D and F make $\measuredangle$s FDE, DFE equal to 2 of the $\measuredangle$s of the given $\triangle$; $\therefore$ $\measuredangle E = 3^d$ angle.

Join AE and produce it to meet the side a in G; from G draw GH, GI, respectively $\|$ to ED, EF.

Join HI. $\triangle HIE$ is the required $\triangle$. ? $\triangle HIG$.
For $\measuredangle E = \measuredangle G$;
also $DE : HG :: AE : AG :: EF : GI$;
$\therefore \triangle HGI \sim \triangle DEF$.

676. $\triangle ABC \sim \triangle A'B'C'$.

677. Corollary. The sect from any angle of a $\triangle$ to H is twice the $\perp$ from O on opposite side.

678. Theorem. Of any $\triangle$, O, ${}^{\mu}C$, and H are co-st'.

Proof. Join OH, meeting AA' in G.
Bisect AG in X and AH in Y.
$XY \parallel$ to and $= \frac{1}{2}GH$, $\therefore \measuredangle AXY = \measuredangle AGH = \measuredangle OGA'$
and $AY = OA'$. [677].
Also $AD \parallel OA'$;
$\therefore XY = OG = \frac{1}{2}GH$,
and $AX = XG = GA'$.
$\therefore G$ is ${}^{\mu}C$.

CO-SYMMEDIAN AND CO-BROCARDAL TRIANGLES.

679. Let the symmedians A_1K, B_1K, C_1K of $\triangle A_1B_1C_1$ be produced to meet the circum-$\odot$ in A_2, B_2, C_2, and let the opposite sides B_1C_1, B_2C_2 of the quad' $B_1C_1B_2C_2$ be produced to meet in L.

Then the polar of L passes through K, the cross of B_1B_2, C_1C_2.

But since A_1A_2 is a symmedian of the $\triangle A_1B_1C_1$, the tangents at B_1 and C_1 cross on A_1KA_2 produced, $\therefore A_1KA_2$ is the polar of L, and the tangents at B_2, C_2 must cross on this st', which is consequently a symmedian of the $\triangle A_2B_2C_2$ also; similarly for the straights B_1KB_2, C_1KC_2.

These $\triangle$s $A_1B_1C_1$, $A_2B_2C_2$ having the same symmedians are called *co-symmedian triangles*.

They have the same Lemoine p't, and the same circumcenter, consequently the same Brocard $\odot$. Also the same

Brocard $\measuredangle$, and the same Brocard p'ts, the same first Lemoine $\odot$, and the same second Lemoine $\odot$, also the Tucker $\odot$s of one are Tucker $\odot$s of the other, though a particular Tucker $\odot$ of one is not always *the same Tucker* $\odot$ of the other. Thus the Taylor $\odot$ of one is not the Taylor $\odot$ of the other.

680. If 2 $\triangle$s are co-symmedian, the sides of one are proportional to the medians of the other.

For $\measuredangle$s $C_2A_2B_2 = C_2A_2A_1 + B_2A_2A_1 = KC_1A_1 + KB_1A_1 = GC_1B_1 + GB_1C_1$, since G and K are isogonal conjugates.

Hence $C_2A_2B_2 = B_1GC_1' = A_1'C_1''G$, where C_1'' is the cross of C_1G with a $\parallel$ to B_1G through A_1'. Similarly, $\measuredangle A_2B_2C_2 = \measuredangle C_1''GA_1'$.

Thus, $\triangle A_2B_2C_2 \sim \triangle C_1''GA_1'$, each of whose sides is $\frac{1}{3}$ the corresponding median of $\triangle A_1B_1C_1$.

681. To show that any $\triangle A_1B_1C_1$ has corresponding to it not only the co-symmedian $\triangle A_2B_2C_2$, but an infinity of others having the same Brocard p'ts, Lemoine p't, Brocard $\odot$, 1st Lemoine $\odot$, 2d Lemoine $\odot$, we should study the triangle's Brocard ellipse, but this would carry us beyond strictly elementary geometry.

A system of $\triangle$s thus connected are called co-Brocardal $\triangle$s.

682. If a $\triangle A_1B_1C_1$ be inscribed in a given $\triangle ABC$, the $\odot$s AB_1C_1, BA_1C_1, CA_1B_1 concur. For let $\odot$s AB_1C_1, BA_1C_1 meet in O.

Then since $B_1OC_1 = \text{st'} \measuredangle - A$, and $C_1OA_1 = \text{st'} \measuredangle - B$,

$$\begin{aligned} \text{we have} \quad B_1OA_1 &= 2 \text{ st'} \measuredangle\text{s} - [\text{st'} \measuredangle - A] - [\text{st'} \measuredangle - B] \\ &= A + B = \text{st'} \measuredangle - C; \end{aligned}$$

$\therefore$ the quad' A_1OB_1C is cyclic.

683. If the $\triangle A_1B_1C_1$ inscribed in $\triangle ABC$ is $\sim$ and co-sensal to it, and A_1 falls on a, then

$$BOC = A + A_1 = 2A;$$

similarly, $COA = 2B$, and $AOB = 2C$;

therefore O is the circumcenter.

684. If A_1 falls on c, the ⊙s concur in Ω.

If A_1 falls on b, the ⊙s concur in Ω'.

In the first case, the △ and its inscribed △ have the same positive Brocard p't.

In the second case, the same negative Brocard p't.

685. Similar △s have the same Brocard ∡ ω.

686. O is the circumcenter of △ ABC; OA_1, OB_1, OC_1 are drawn to a, b, c, so that ∡ OA_1A = ∡ OB_1B = ∡ OC_1C.

Show that △ $A_1B_1C_1$ is ∼ and co-sensal to △ ABC.

Show also that O is the orthocenter of △ $A_1B_1C_1$.

EXERCISES ON BOOK VIII.

1. Through the mid point of each side of a △ are drawn ⊥s to the other 2 sides. Show that the 2 △s thus formed have the same Lemoine p't.

2. Show that the Lemoine p't of a △ is the centroid of its ⊥ projections on the sides; and inversely.

3. Find a p't within a △ such that the sum of the squares of its ⊥s on the sides is a minimum.

[The p't must be the centroid of its ⊥ projections on the sides, and ∴ the Lemoine p't.]

4. The joins of the circumcenter of a △ to its vertices are ⊥ to the sides of its orthocentric △.

5. Brocard's first △ is in perspective with its original △.

6. Brocard's first △ is ∼ but not cosensal to its original.

7. The join of the circumcenter and Lemoine p't of a △ is the r't bi' of the join of its Brocard p'ts.

$$[\measuredangle\ \Omega\alpha K = \measuredangle\ \Omega BC = \omega = \Omega' CB = \Omega'\alpha K.]$$

8. If T is the center of Brocard's ⊙, then $\measuredangle\ \Omega T\Omega' = 2\Omega TK = 4\omega$.

9. If AK, BK, CK meet the Brocard ⊙ in α', β', γ', then α' is the cross of the Brocard arcs $A\Omega C$, $A\Omega' B$; β' is the cross of $B\Omega A$, $B\Omega' C$; and γ' is the cross of $C\Omega B$, $C\Omega' A$.

10. [Dewulf.] If through the Brocard p't Ω three ⊙s be described each passing through two vertices of △ ABC, the △ formed by their centers has the circumcenter of ABC for one of its Brocard p'ts.

11. The join of any two p'ts, and the join of their isogonal conjugates with respect to a △, subtend at any vertex of the △ ∡ either = or supplemental.

12. If three st's through the vertices of a △ meet the opposite sides co-st'ly, so do their isogonals.

13. The joins of the ⊥ projections of the Lemoine p't on the sides of the △ are ⊥ to the medians.

14. If on a given sect, and on the same side of it, be described six △s ∼ to a given △, the vertices are concyclic.

15. In Fig. 275, $A\alpha$, $B\beta$, $C\gamma$ concur.

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